On proving, and on writing proofs

On assignments you are asked to “show that …” or “prove that …”. To do so you must clearly understand the full range of cases you are addressing. You must understand what assumptions you may make and should make. And you must understand the conclusion you wish to draw. (Looking at some particular cases may be they way to get these understandings.) Then you must make an appropriately general, precise, and complete argument which shows that your assumptions imply your conclusion. That is, you need to *prove*. A proof is a careful argument that reflects complete logical understanding of a situation.

For example, suppose an exercise says:

> **Exercise 666.** Show that if A is an invertible $m \times m$ matrix and if B is an $m \times n$ matrix of full rank, with $n \leq m$, then AB has full rank.

An appropriate **solution** starts with a statement (restatement) of what is proved:

> Suppose $m \geq n$. Suppose $A \in \mathbb{C}^{m\times m}$ is an invertible matrix and $B \in \mathbb{C}^{m\times n}$ is a matrix with full rank. If $C = AB$ then C has rank n, which is to say full rank.
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> *Proof.* Note that $C \in \mathbb{C}^{m\times n}$. Let v_1, v_2 be distinct vectors in $\mathbb{C}^n$. By Theorem 1.2 in Trefethen & Bau, because B has full rank, $w_1 = Bv_1$ and $w_2 = Bv_2$ are distinct vectors in $\mathbb{C}^m$. By Theorem 1.3, A has full rank, so by Theorem 1.2 $z_1 = Aw_1$ and $z_2 = Aw_2$ are distinct vectors. But
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> $$z_i = Aw_i = A(Bv_i) = (AB)v_i = Cv_i$$
>
> for $i = 1, 2$. Thus C maps distinct vectors v_1, v_2 to distinct vectors z_1, z_2. Again by Theorem 1.2, C has full rank. □

Note these style elements:

- What I assume is clearly stated. Do not be afraid to restate the exercise.
- What I intend to prove (i.e. the claim “C has rank n”) is clearly stated.
- The proof is separated from the claim, and its beginning and end are indicated.

Such a concrete style helps when I am determining if your argument does or does not show/prove the claim. It also helps you. For instance, if you find you cannot prove the most general statement, but you can prove something which (for instance) has stronger assumptions but the same conclusion, then that situation is *clear*. And you will get an appropriate amount of credit. I will give much less credit for a confused statement of what has been proved.

I recommend the style of proof used here. You are not obliged to use it, but you must still make the careful and complete argument.