# Review Topics for In-Class Midterm Quiz on *Friday 29 March, 2013*

The Midterm Quiz will cover Lectures 1, 2, 3, 4, 5, 6, 7, 8, 10, 11 in Trefethen & Bau. The problems will be of these types: state definitions, state theorems, state algorithms (as pseudocode or Matlab), describe or illustrate basic geometrical ideas, make basic applications of theorems, make basic calculations, prove simple theorems/corollaries.

**Definitions**. Know how to define:

- matrix-vector product; matrix-matrix product
- adjoint; hermitian; transpose
- inner product; outer product
- unitary
- $\|\cdot\|_p$ of a vector in $\mathbb{C}^m$ for any $1 \le p \le \infty$
- induced matrix norm $\|\cdot\|$
- Frobenius matrix norm $\|\cdot\|_F$
- projector; orthogonal projector
- eigenvalue; eigenvector; singular value

**Matrix Factorizations and Constructions**. Here $A \in \mathbb{C}^{m\times n}$ unless otherwise stated. Know the properties of the factors. (E.g. know that "$\hat{U}$ has orthonormal columns and is of same size as $A$ in reduced SVD factorization $A = \hat{U}\hat{\Sigma}V^*$.") Be able to use the factorization in simple calculations. Be able to compute the factorization by hand in sufficiently simple cases.

- full SVD: $A = U\Sigma V^*$
- reduced SVD, $m \ge n$: $A = \hat{U}\hat{\Sigma}V^*$
- full QR, $m \ge n$: $A = QR$
- reduced QR, $m \ge n$: $A = \hat{Q}\hat{R}$
- eigenvalue, $m = n$: $AX = X\Lambda$ and $X$ is invertible
- orthogonal projector, $\hat{Q}$ has $m \ge n$: $P = \hat{Q}\hat{Q}^*$
- orthogonal projector, $A$ full rank with $m \ge n$: $P = A\,(A^*A)^{-1}\,A^*$
- Householder reflector: $F = I - 2vv^*/(v^*v)$ and $Q = \left[\begin{smallmatrix} I & 0 \\ 0 & F \end{smallmatrix}\right]$

**Algorithms**. Be able to state these algorithms, including the amount of work to leading order:

- 7.1: classical Gram-Schmidt for $QR$
- 8.1: modified Gram-Schmidt for $QR$ [*informal knowledge o.k.*]
- 10.1: Householder triangularization for $QR$
- 10.2: compute $Q^*b$ after Householder triangularization
- 11.2: $QR$ "modern classical" for least squares (for overdetermined "$Ax = b$")
- 11.1: normal equations for least squares (for overdetermined "$Ax = b$")

**Facts, Formulas, and Inequalities**. Know these as facts. Be able to prove unless otherwise stated.

- Cauchy-Schwarz: $|x^*y| \leq \|x\| \, \|y\|$ *[proof not required]*
- 1-norm of a matrix is the maximum absolute column sum
- $\infty$-norm of a matrix is the maximum absolute row sum
- invariance of $\|\cdot\|_2$ and $\|\cdot\|_F$ matrix norms under unitaries
- $\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2} = \sqrt{\operatorname{tr}(A^*A)}$
- $\|A\|_2 = \sigma_1 = \sqrt{\rho(A^*A)}$
- $\operatorname{rank}(A)$ is number of nonzero singular values (in exact arithmetic)
- for $A \in \mathbb{C}^{m\times n}$, $A$ has full rank if and only if $A^*A$ is nonsingular
- the singular values of $A$ are the square roots of the eigenvalues of $A^*A$
- if $P$ is a projector then so is $I - P$
- if $P$ is an orthogonal projector then so is $I - P$, and furthermore $I - 2P$ is unitary

**Ideas**. In this class there are ideas to be comfortable with! In some cases in the list below there are provable theorems, but in other cases there is just a perspective or paradigm to understand:

- L1 and L2: how to think about $Ax$, $A^{-1}b$, $Qx$, $Q^*b$
- L4: the image of the unit sphere under any $m \times n$ matrix is a hyperellipse
- L5: sums like this are optimal (in what sense?) approximators of $A$:

$$A_\nu = \sum_{j=1}^{\nu} \sigma_j u_j v_j^*$$

- construction of orthogonal functions (e.g. orthogonal polynomials) is instance of Gram-Schmidt and of "$QR$" where columns are infinitely long
- the asymptotic number of operations in an algorithm can be counted by counting only number of times inner-most operations are executed
- the number of operations in an algorithm can be counted by geometric arguments
- Gram-Schmidt is "triangular orthogonalization" while Householder is "orthogonal triangularization"