Assignment # 8

Due Friday 12 April, 2013

Please read Lectures 12, 13, 14, and 15 in the textbook *Numerical Linear Algebra* by Trefethen and Bau. Do these exercises:

Exercise 12.1 in Lecture 12.

Exercise 13.2ab in Lecture 13.

Exercise 13.3 in Lecture 13.

Exercise 14.1fg in Lecture 14.

Exercise 14.2 in Lecture 14.

P16. Reproduce figures 11.1 and 11.2 on pages 79 and 80 of the textbook. Show the code you used. (*Hint: Type* `help vander`.)

P17. A *circulant matrix* is one where the diagonals are constant and "wrap around":

$$C = \begin{bmatrix} c_1 & c_m & \dots & c_3 & c_2 \\ c_2 & c_1 & c_m & & c_3 \\ \vdots & c_2 & c_1 & \ddots & \vdots \\ c_{m-1} & & \ddots & \ddots & c_m \\ c_m & c_{m-2} & \dots & c_2 & c_1 \end{bmatrix} \tag{1}$$

The entries of $C \in \mathbb{C}^{m\times m}$ are a function of the difference of the row and column indices, mod m:

$$C_{jk} = \begin{cases} c_{j-k+1}, & j \geq k, \\ c_{m+j-k+1}, & j < k. \end{cases}$$

Here $c_1, \dots, c_m$ can be regarded as the entries of a column vector, the first column of C. Specifying the first column of a circulant matrix describes it completely.

Download this MATLAB function, which builds a circulant matrix with a given first column; notice how it uses the `mod()` function:

`http://www.dms.uaf.edu/~bueler/circu.m`

(a) Recall that $\{e_j\}_{j=1}^m$ denotes the standard basis for $\mathbb{C}^m$. Use `circu.m` to generate circulant matrices C_1 and C_2 which have e_1 and e_2, respectively, for their first columns. Note C_1 is just the identity. Explain why C_2 is the "downshift" matrix. What is the inverse of C_2?

(b) Define the *periodic convolution of vectors* $u, w \in \mathbb{C}^m$ by

$$(u * w)_j = \sum_{k=1}^{m} u_{\mu(j,k)} v_k \qquad \text{where} \qquad \mu(j,k) = \begin{cases} j - k + 1, & j \geq k, \\ m + j - k + 1, & j < k. \end{cases}$$

Show, just to exercise this notation, that $u * w = w * u$.

(c) Show that $Cu = v * u$ if C is a circulant matrix and v is the first column of C.

(d) Here is an extraordinary fact about circulant matrices: Every circulant matrix has a complete set of eigenvectors that are known in advance, without knowing the eigenvalues. Specifically, define $f_k \in \mathbb{C}^m$ by

$$(f_k)_j = \exp\left(-i(j-1)(k-1)\frac{2\pi}{m}\right) = e^{-i2\pi(k-1)(j-1)/m},$$

where, as usual, $i = \sqrt{-1}$. Convince yourself that these vectors are *waves*, i.e. combinations of familiar sines and cosines. In particular, choose $m = 100$ and $k = 0, 1, 2, 3, 49$ and clearly plot the real and imaginary parts of f_k. Show that for any m, the eigenvectors $\{f_1, \dots, f_m\}$ are orthogonal.

(e) Now, for the general circulant matrix C in (1) above, confirm the "extraordinary fact" as follows: Give a formula for the eigenvalues λ_k, in terms of the entries $c_1, \dots, c_m$, by showing by direct by-hand calculation that $Cf_k = \lambda_k f_k$.

(Extra Credit) Consider a constant-coefficient linear ODE, of possibly high order, on an interval with periodic boundary conditions. The ODE might be written "$(\mathcal{L}y)(x) = f(x)$" for a constant-coefficient linear differential operator $\mathcal{L}$. Consider applying a finite difference discretization, using an equally-spaced mesh with m spaces. With appropriate clear writing, explain how this gives a circulant matrix problem. Answer the question: What is the highest-frequency wave that the grid with m spaces can support? Then explain why the conclusion in (e) is not surprising in this case.

(Just for fun) Before running this single line of MATLAB, predict what will happen:

```
C=circu(randn(25,1)); [V,D]=eig(C); plot(V), axis equal, axis off
```

Now run it, several times. Explain. Groovy!

At some point it may be useful to see the Wikipedia pages for "circulant matrix," "discrete Fourier transform," and "DFT matrix." No secret answers are revealed, but it may help you organize thoughts.