Steepest descent

is not great

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MATH 661 Optimization

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steepest descent for unconstrained optimization

- for the day I am away, here is a distraction from linear programming
- these slides are a brief introduction to a well-known topic in unconstrained optimization, namely ...
- steepest descent
  - a.k.a. gradient descent
- the textbook[1] puts it off till later but you should be aware of it now
  - please read sections 12.1 and 12.2, but ignore the Lemmas for now

[1] Griva, Nash & Sofer, *Linear and Nonlinear Optimization*, 2nd ed., SIAM Press 2009

why you should know about steepest descent

- if you deal with optimization in the real world you will see it as a proposed algorithm for stuff
- for easy problems it is the lazy-person's algorithm
    - "easy" roughly means:
        - smooth
        - dimension $< 10^6$ (or so)
        - unconstrained
    - I don't recommend steepest descent
    - ...but it might minimize total programmer time
- for hard problems it may be the only thing you can implement
    - e.g. big machine learning problems, big nonlinear inverse problems, ...
    - a version of steepest descent may be the standard in your industry
    - e.g. *stochastic gradient descent* is a nice, popular buzzword
        - it's even slower than ordinary steepest descent

the steepest descent algorithm

- assume $f : \mathbb{R}^n \to \mathbb{R}$ has (at least) one continuous derivative
- we want to solve the unconstrained problem:

$$\min_{\mathbb{R}^n} f(x)$$

- the algorithm:

  1. User supplies x_0.
  2. For $k = 0, 1, 2, \dots$
     (i) If x_k is optimal then stop.
     (ii) Search direction is
     $$p_k = -\nabla f(x_k)$$
     (iii) Determine step length $\alpha_k > 0$. Let $x_{k+1} = x_k + \alpha_k p_k$.

steepest descent is obvious

- it is an *obvious* interpretation of general optimization algorithm II in §2.4
  - direction is chosen as “go straight downhill”
    - recall from calculus: the gradient points straight uphill
  - but we don’t know how to use the length of $\nabla f(x_k)$
  - ... so we *must* make a choice for α_k
  - also we need a stopping criterion
- *any* choice of steepest descent length, i.e. $p_k = -c\nabla f(x_k)$ and $c > 0$, generates a (feasible) descent direction at x_k
  - recall: p is a *descent direction at x* if $p^\top \nabla f(x) < 0$
- choosing $p_k = -\nabla f(x_k)$ uses the direction which solves this directional-derivative optimization problem

$$\min_{\|q\|=1} q^\top \nabla f(x_k)$$

one way to choose step length: back-tracking

- we will see in section 11.5 that we can prove convergence of many unconstrained optimization algorithms as long as the step-size α_k is chosen to satisfy certain conditions
- for now I just need *some* reasonable way to choose α_k
- the standard way to satisfy these conditions is called "back-tracking"
    - page 378 of the textbook
    - an implementation:

```
function alpha = bt(xk,pk,dfxk,f)
Dk = dfxk' * pk;      % scalar directional derivative; negative
c = 1.0e-4;           % modest sufficient decrease
rho = 0.5;            % backtracking by halving
alpha = 1.0;
while f(xk + alpha * pk) > f(xk) + c * alpha * Dk
    alpha = rho * alpha;
end
```

    - we will return to this topic

steepest-descent-back-tracking code

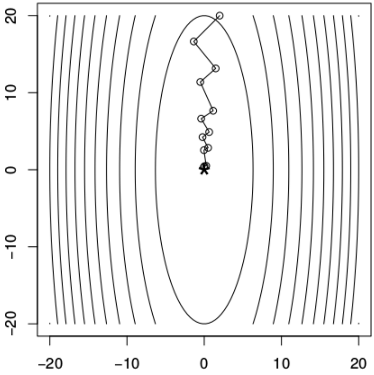
- here is a basic implementation of *steepest-descent-with-back-tracking* $=$ SDBT
- it assumes that the user supplies x_0 and a function f that returns both the values $f(x)$ and the gradient $\nabla f(x)$:

```
xk = x0;
for k = 1:maxiters
    [fxk, dfxk] = f(xk);
    if norm(dfxk) < tol
        break
    end
    pk = - dfxk;                  % steepest descent
    alpha = bt(xk,pk,dfxk,f);     % back-tracking
    xk = xk + alpha * pk;
end
```

- you can set `maxiters` to 10^4 or so to avoid long waits for failure

steepest-descent-back-tracking: example I

- suppose $f(x) = 5x_1^2 + \frac{1}{2}x_2^2$ for $x \in \mathbb{R}^2$, an easy quadratic objective function with global minimum at $x^* = (0,0)^\top$
- result from SDBT:



- is this result o.k.?

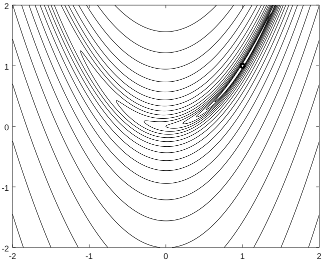
steepest-descent-back-tracking: example II

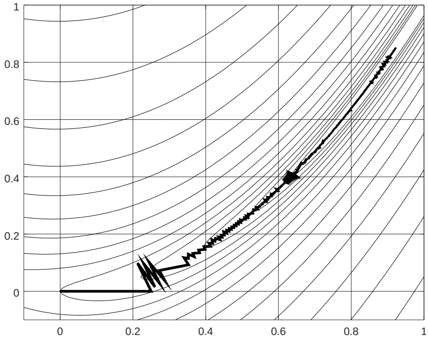
- a famously-harder problem in $\mathbb{R}^2$ is to minimize the *Rosenbrock function*:

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

  - is a *quartic* polynomial in 2 variables
  - has a single global minimum at $x^* = (1,1)^\top$
  - has steep "banana" shaped contours (bottom left)

- at right: SDBT from $x_0 = (0,0)^\top$
  - struggles





quadratic functions

- now we consider steepest descent for quadratic functions in $\mathbb{R}^n$
- such functions can always be written

$$f(x) = \frac{1}{2} x^\top Q x - c^\top x + d$$

  - Q is a symmetric square matrix, c is a column vector, $d \in \mathbb{R}$
  - exercise **P9** on A5: check that

$$\nabla f(x) = Q x - c$$

  - *assume* Q is positive definite
    - then f is strictly convex
    - and there is unique global (and local) minimizer where $\nabla f = 0$: $\qquad x^* = Q^{-1} c$
  - the additive constant d can be ignored in optimization problems because it neither affects $\nabla f(x)$ nor the location of x^*
  - example 1: $c = 0$; Q is a diagonal 2×2 matrix with $5, 1/2$ on diagonal

line search for quadratic functions

- given any descent direction p_k at x_k, the *optimal* step size is

$$\alpha_k = \frac{-p_k^\top \nabla f(x_k)}{p_k^\top Q p_k} = \frac{p_k^\top (c - Q x_k)}{p_k^\top Q p_k}$$

  - showing this is in exercise **P9** on A5
- this α_k minimizes $g(\alpha) = f(x_k + \alpha p_k)$ over $\alpha > 0$
- thus back-tracking is *not* needed for quadratic functions

- but steepest descent is still slow
  - exercise **P10** on A5 asks you to reproduce Example 12.1 in section 12.2 of the textbook
  - steepest descent with optimal step size uses a totally-unnecessary 216 steps to get modest accuracy
  - since we have the optimal step size α_k, the problem in steepest descent must be that the steepest descent direction is wrong

steepest descent is the wrong direction

- for quadratic objective functions $f(x) = \frac{1}{2}x^\top Qx - c^\top x$, where the gradient is a linear function, the Newton iteration converges to $x^* = Q^{-1}c$ in one step
- Newton uses this direction:

$$p_k = -\left(\nabla f(x_k)^\top\right)^{-1} \nabla f(x_k)$$

- steepest descent uses:

$$p_k = -\nabla f(x_k) = -(I)^{-1}\nabla f(x_k)$$

- unconstrained optimization usually benefits a lot from using the information in the Hessian to turn away from the steepest descent direction $-\nabla f(x_k)$
    - that's why we will care about the rest of Chapters 11, 12, and 13
    - especially "quasi-Newton" methods

summary slide

- steepest descent simply uses search direction $p_k = -\nabla f(x_k)$
- determining the step size α_k is nontrivial
  - line search (section 11.5) or trust region (11.6) is needed
  - for general functions, back-tracking is recommended
  - for quadratic functions we can use the optimal step size
- even with good line search, steepest descent sucks
  - steepest descent is slow when contour lines (level sets) are highly curved
  - going down the gradient is generally the wrong direction
- for quadratic functions Newton is clearly better: one step convergence
- hard functions like Rosenbrock are hard even for Newton