Review Guide for in-class Midterm Exam on Friday, 26 October 2018

The in-class Midterm Exam will cover the sections listed below from Griva, Nash, Sofer, *Linear and Nonlinear Optimization*, 2nd ed., 2009. It will only cover topics that have appeared on homework and in lecture. You are *not* responsible for any material in Chapters 7, 8, 9, 10, 13, 14, 15, or 16.

The Exam is *closed book* and *no calculator*. You may bring your own notes on *half* of a single sheet of letter-sized paper.

The problems will be of these types: state definitions, state or prove certain theorems (see below), prove or show propositions which follow directly from the definitions or from known facts, give examples with certain properties, describe or sketch concepts, or compute a step or two of certain algorithms.

I encourage you to get together with other students and work through this Review Guide. Be honest with yourself about what you do and don't know, and talk it through and learn!

The list of Definitions below has blanks for page numbers. Use it as a "worksheet;" you can study by finding the page where the term is defined. On Wednesday 10/24 I will post "solutions" for these page numbers at the website.

Sections. You are *only* responsible for material in these Sections:

- 1.1–1.5
- 2.1–2.7
- 3.1, 3.2
- 4.1–4.4
- 5.1, 5.2 (but *not* 5.2.3, 5.2.4)
- 6.1 (*only* page 177), 6.2 (but *not* 6.2.2)
- 11.1–11.3
- 12.1, 12.2 (but *not* the Lemmas in 12.2)
- Appendices A.1–A.5, A.7.1
- Appendices B.4, B.5.

Definitions. Be able to define the term (word or phrase). Understand and/or use the term correctly. Be able to prove things that follow immediately from the definition:

- *transpose* $A^\top$ of an $m \times n$ matrix A p. ______
- *symmetric* matrix p. ______
- *nonsingular* matrix p. ______
- *positive definite* matrix p. ______
- *gradient* $\nabla f(x)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ p. ______
- *gradient* $\nabla f(x)$ of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ p. ______
- *Jacobian* $\nabla f(x)^\top$ of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ p. ______
- *Hessian* $\nabla^2 f(x)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ p. ______

- *feasible set* $S \subseteq \mathbb{R}^n$, e.g. as defined by constraints on page 43 p. ______
- *feasible point* p. ______
- *active* constraint at $x \in S$ p. ______
- *inactive* inequality constraint at $x \in S$ p. ______
- *interior* point of S p. ______
- *global minimizer* of f in S p. ______
- *strict global minimizer* p. ______
- *local minimizer* of f in S p. ______
- *strict local minimizer* p. ______
- *convex* set S p. ______
- *convex* function f on a convex set S p. ______
- *strictly convex* function p. ______
- *convex* optimization problem p. ______
- *convex combination* of a finite set of points in $\mathbb{R}^n$ p. ______
- *search direction* in an optimization algorithm p. ______
- *step length* p. ______
- *descent direction* p. ______
- *feasible direction* p. ______
- *feasible descent direction* p. ______
- *line search* p. ______
- *linear (rate of) convergence* for a convergent sequence p. ______
- *superlinear (rate of) convergence* p. ______
- *quadratic (rate of) convergence* p. ______
- *null space* of an $m \times n$ matrix A p. ______
- *range space* of A (or $A^\top$) p. ______
- *orthogonal subspaces* p. ______
- *null space matrix* of A p. ______
- *standard form* of a linear programming problem (l.p.p.) p. ______
- *free variable* p. ______
- *slack variable* p. ______
- *excess variable* p. ______
- *extreme point* (or *vertex*) of a convex set S p. ______
- *basic solution* of a standard form l.p.p. p. ______
- *basic feasible solution* of a standard form l.p.p. p. ______
- *optimal basic feasible solution* of a standard form l.p.p. p. ______
- *degenerate vertex* of a standard form l.p.p. p. ______
- *unbounded direction* of a standard form l.p.p. p. ______
- *dual problem* for a standard form l.p.p. p. ______
- *primal problem* p. ______
- *stationary point* p. ______
- *first-order necessary conditions* p. ______
- *second-order necessary and sufficient conditions* p. ______

Theorems. Understand these theorems, and be able to use them as facts. Be able to illustrate with an example or a sketch. Be able to prove those that say so.

- Theorem 2.1 (global solutions of convex optimization problems) p. 49
- characterizations of convexity: p. 51
  - f is convex if it is above its tangent planes: $f(y) \geq f(x) + \nabla f(x)^\top (y - x)$
  - f is strictly convex if its Hessian $\nabla^2 f(x)$ is positive definite for all x
- Taylor series and remainder form in one dimension pp. 64–65
- Taylor series and remainder form in n dimensions, to $O(p^2)$ term pp. 64–65
- Theorem 4.4 (extreme point $\iff$ basic feasible solution) p. 110
- Theorem 4.6 (representation of x as convex combination of b.f.s.) p. 120
- Theorem 4.7 (x optimal $\implies$ there is optimal b.f.s.) p. 121
- Theorem 5.10 (simplex method terminates) ***be able to prove*** p. 163
- Theorem 6.4 (weak duality) ***be able to prove*** p. 179
- Corollaries 6.6 and 6.7 ***be able to prove*** p. 179
- Theorem 6.9 (strong duality) p. 180
- Theorem 6.11 (complementary slackness) p. 183
- Theorem 11.2 (quadratic convergence of Newton's method) p. 366

Theorem about feasible directions for linear constraints. For clarity I state these ideas from Section 3.1 separately. Understand them and be able to use them as facts. Be able to illustrate with an example or a sketch. Be able to prove them.

- notation for constraints:

  $$\mathcal{E} \text{ is index set for } a_i^\top x = b_i, \qquad \mathcal{I} \text{ is index set for } a_i^\top x \geq b_i$$

  and $\hat{\mathcal{I}}$ denotes active inequality constraints at a given point $\bar{x}$
- p is a feasible direction at feasible point $\bar{x}$ if and only if p. 80

  $$a_i^\top p = 0 \text{ for } i \in \mathcal{E} \qquad \text{and} \qquad a_i^\top p \geq 0 \text{ for } i \in \hat{\mathcal{I}}$$
- given a feasible direction p, only the inactive inequality constraints are relevant when determining an upper bound on step length α p. 81
- if $a_i^\top p \geq 0$ for all inactive inequality constraints then an arbitrarily-large step can be taken while staying feasible; there is no upper bound on α p. 81
- otherwise the maximum allowed step length arises from the *ratio test*: p. 81

  $$\bar{\alpha} = \min \left\{ \frac{(a_i^\top \bar{x} - b)}{-a_i^\top p} \; : \; i \text{ is inactive at } \bar{x} \text{ and } a_i^\top p < 0 \right\}$$

Algorithms. Be able to state the algorithm as a pseudocode. Be able to apply/execute it in simple cases.

- General Optimization Algorithm II — p. 55
- Newton's method for a single equation in one variable — p. 67
- Newton's method for n equations in n unknowns — p. 73
- rules for converting l.p.p. to standard form (be able to *apply* them) — pp. 101–102
- simplex method for l.p.p. in standard form — p. 131
- Newton's method for minimization — pp. 364–365
- steepest descent method (if given step lengths) — p. 403