Assignment #7

REVISED DATE $\to$ Due Wednesday, 16 November at the start of class

Please read sections 10.1, 10.2, 10.3, 11.1, and 11.2 in Nocedal & Wright. Do the following Exercises and Problems. (*This is a deliberately short assignment which does the minimum to help you read the material.*)

Exercise 10.1 (a). (*Hints*: If $m \geq n$ and $J \in \mathbb{R}^{m\times n}$ then J has full column rank if and only if $Jv = 0$ implies $v = 0$. Note that Jv is a linear combination of the columns of J. Also, $A \in \mathbb{R}^{n\times n}$ is nonsingular if and only if $Av = 0$ implies $v = 0$.)

Exercise 10.2. (*Hint*: One way to do this is to use the result of Exercise 2.7, which you have already done.)

Exercise 11.4. (*Hint*: The sum-of-squares merit function is (11.35).)

Problem P19. *This problem does concrete, by-hand calculations for a Chapter 10 problem, i.e. nonlinear least squares, in a case where $m = 4$ and $n = 2$.*

(a) Consider the data:

$$\begin{aligned} t_1 &= 0 & \qquad y_1 &= 3 \\ t_2 &= 1 & y_2 &= 1 \\ t_3 &= 2 & y_3 &= 2 \\ t_4 &= 3 & y_4 &= 1 \end{aligned}$$

Let $x = (x_1, x_2) \in \mathbb{R}^2$ be the parameters. Our model is the function

$$\phi(x; t) = x_1 e^{x_2 t}.$$

Compute the quantities $r_j(x)$, $r(x)$, and $J(x)$. They are all defined at the beginning of Chapter 10, namely in equations (10.8), (10.2), and (10.3). Please simplify these quantities as far as possible.

(b) Show that $J(x)$ in part **(a)** has full rank if and only if $x_1 \neq 0$.

(c) Suppose $x^{(0)} = (2, 0)$ is the initial iterate in the Gauss-Newton algorithm. Compute the next iterate $x^{(1)}$ assuming that the full step is used (i.e. $\alpha_k = 1$ from the line search). Start by expressing the linear system (10.23) in form $Ap = b$ where A, b are fully-simplified; please show A and b. Then use MATLAB to compute p and $x^{(1)}$.

(d) Suppose we accept $x^{(1)}$ from **(c)** as an adequate solution to the problem. Plot the resulting curve on top of the data.

Problem P20. *This problem does concrete, by-hand calculations for a Chapter 11 problem.*

(a) Consider the system of equations

$$x^2 + y^2 = 1$$
$$y = \frac{1}{2}e^{2x}.$$

Put this system in the form (11.1), namely $r(x) = 0$. Compute the sum-of-squares merit function $f(x)$ in (11.35). Also, give a sketch which illustrates that there are two solutions $x \in \mathbb{R}^2$, and shows roughly where these solutions are.

(b) Show that the equations $r(x) = 0$ in part **(a)** are *not* of the form "$\nabla g(x) = 0$" for any smooth scalar function $g(x)$. (*Hint*. Symmetry of a matrix.)

(c) Consider the line-search Newton method, Algorithm 11.4. Let $x_0 = (1, 1)$ be the first iterate. Compute p_0. Then, given that the line search solves

$$\min_{\alpha>0} \phi(\alpha) = \min_{\alpha>0} f(x_k + \alpha\, p_k),$$

where $f(x)$ is the merit function, use MATLAB to plot $\phi(\alpha)$ on an appropriate interval. This graph should show the location of the exact line search minimum.

(d) Does the full Newton step $\alpha_k = 1$ occur before or after the exact line search minimum? With $c_1 = 10^{-4}$ and $c_2 = 1/4$, does the full Newton step $\alpha_k = 1$ in part **(c)** satisfy the Wolfe conditions?