Final Exam

100 points

Due at 12:00 NOON on THURSDAY DECEMBER 20, 2001

Rules. You may use any printed reference materials, but you should reference them clearly. You may and should talk to me for help. You may **not** talk to each other or to any other people, including electronically. Please write your solutions clearly and start each solution by stating what you are proving. At least one page per problem. Thanks!

1. (10 points) Section 2.6 #48 in Royden.

2. (10 points) Section 5.4 #15.

3. (10 points) Section 5.2 #8. [Read section 5.2 for this and the next problem. Section 5.2 and these two problems do not depend logically on section 5.1, or any particular topic, actually.]

4. (10 points) Section 5.2 #10.

5. (10 points) Construct a sequence of nonnegative measurable functions $f_n : [0,1] \to \mathbf{R}$ so that $\limsup_{n\to\infty} f_n(x) = +\infty$ for each $x \in [0,1]$ but for which

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0.$$

[*Hint*: It can be done with step functions.]

6. (10 points) Let $f : [0,1] \to \mathbf{R}^*$ be integrable. Suppose that for every continuous function $g \in C([0,1])$,

$$\int_0^1 f(x)g(x)\,dx = 0.$$

Prove that $f = 0$ a.e.

[Sergei asked in class what $\mathcal{M}^\perp$ is when $\mathcal{M} = C([0,1])$ as a subspace of the Hilbert space $L^2([0,1])$. The above says $\mathcal{M}^\perp = \{0\}$ directly—without talking about ${\mathcal{M}^\perp}^{\perp\perp}$ . . .]

7. (10 points) Prove that the supremum of any collection of convex functions on (a,b) is convex on (a,b) (if it is finite). Prove that pointwise limits of sequences of convex functions are convex.

8. (10 points) Let M be the set of all $f \in L^1([0,1])$ such that

$$\int_0^1 f(t)\,dt = 1.$$

Show that M is a closed convex subset of $L^1([0,1])$ which contains infinitely many elements of minimal norm.

[That is, show that if a sequence in M converges then it converges to an element of M. Then show that if $f, g \in M$, it follows that $\lambda f + (1-\lambda) f \in M$. Finally, determine the minimum norm for elements of M and (concretely) find infinitely many elements of M with this norm.]

9. (10 points) Show that there is a bounded linear functional F on $L^\infty([0,1])$ that is not identically zero but does give zero on $C([0,1])$.

[Such an example turns out to be why we did not allow $p = \infty$ in the Riesz Representation Theorem:]

Extra Credit. (4 points) Use the previous problem and a theorem in real analysis which we did not prove to show that the dual space of $L^\infty([0,1])$ is *not* isometrically isomorphic to $L^1([0,1])$—even though $p = \infty$ and $q = 1$ are conjugate.

10. (10 points) Prove a generalized Hölder inequality: Suppose $p, q, r \geq 1$ satisfy $p^{-1} + q^{-1} = r^{-1}$. Suppose $f \in L^p([0,1])$, $g \in L^q([0,1])$. Then $fg \in L^r([0,1])$ and

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

[*Hint*: p/r and q/r are conjugate.]