Final Exam

100 points

Due at 12:00 NOON on WEDNESDAY DECEMBER 20, 2000

Rules. You may use any printed reference materials, but you should reference them clearly. You may and should talk to me for help. You may **not** talk to each other. Please write your solutions clearly and start each solution by restating the problem. At least one page per problem. Thanks!

1. (10) (Section 5.4 #18:) Let g be an absolutely continuous function on $[0,1]$ and $E \subset [0,1]$ a set of measure zero. Then $g(E)$ has measure zero.

2. (10) Suppose f_n is a sequence of bounded measurable functions on $[a,b]$ and $f_n \to f$ uniformly on $[a,b]$. Prove that

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

3. (10) Construct a sequence of continuous functions f_n on $[0,1]$ such that $0 \le f_n \le 1$ and such that

$$\lim_{n\to\infty} \int_0^1 f_n = 0,$$

but such that the sequence $\{f_n(x)\}$ converges for no $x \in [0,1]$. [*Hint: It may be easier to construct a sequence of step functions with this property, and then modify the elements of that sequence so that the functions are continuous.*]

4. (5) Let $\{\varphi_n\}_{n=1}^{\infty}$ be a complete (maximal) orthonormal set on $L^2[0,\infty)$. Show that

$$\sum_{n=1}^{\infty} \left| \int_0^x \varphi_n(t)\,dt \right|^2 = x.$$

5. (10) Define the *essential range* of a function $f \in L^\infty(E)$ to be the set R_f of all w such that

$$m(\{x : |f(x) - w| < \epsilon\}) > 0$$

for every $\epsilon > 0$. Prove that R_f is compact. Also give the relation between the set R_f and the number $\|f\|_\infty$.

6. a. (5) Define $u_s(t) = e^{ist}$ for all $s, t \in \mathbf{R}$. Let X be the complex vector space of those functions $f(t)$ on $\mathbf{R}$ which are finite linear combinations of functions $u_s(t)$. If $f, g \in X$ define

$$(f, g) = \lim_{A \to \infty} \frac{1}{2A} \int_{-A}^{A} \overline{f(t)} g(t)\, dt.$$

Show that X is an inner product space.

b. (5) Show that the set $S = \{u_s : s \in \mathbf{R}\}$ is an orthonormal set.

7. a. (5) Suppose f is a measurable function on A, assume $\|f\|_\infty > 0$, and define

$$\varphi(p) = \int_A |f|^p = \|f\|_p^p, \qquad 0 < p < \infty$$

Let $E = \{p : \varphi(p) < \infty\}$. If $r < p < s$ and $r, s \in E$, prove that $p \in E$.

b. (5) Prove that $\log \varphi$ is convex and that φ is continuous on E.

c. (5) If $r < p < s$, prove that $\|f\|_p \leq \max\{\|f\|_r, \|f\|_s\}$.

8. (10) If M is a subspace of a Hilbert space $\mathcal{H}$, show that $\overline{M} = \left(M^\perp\right)^\perp$. (Note: $\overline{X}$ is the closure of X, that is, the smallest closed set that contains X.)

9. a. (5) Show that the unit ball $B = \{x : \|x\| \leq 1\}$ in any normed vector space is *convex*, that is, if $x, y \in B$ then $\lambda x + (1 - \lambda) y \in B$ for every $0 \leq \lambda \leq 1$.

b. (10) Show that if $1 < p < \infty$ the unit ball of $L^p(E)$ is *strictly* convex, that is, if $\|f\|_p = 1$ and $\|g\|_p = 1$, $f \neq g$, and $h = \frac{1}{2}(f + g)$, then $\|h\|_p < 1$.

10. (5) Prove that

$$\lim_{n \to \infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx = 1.$$

Commentary

Remark on 1. This implies that the Cantor ternary function f—see problem 2.48, page 50—is not absolutely continuous!

Remark on 6. Let $\mathcal{H}$ be the completion of X, so $\mathcal{H}$ is the Hilbert space of equivalence classes of Cauchy sequences in X, under the equivalence that the distance between two Cauchy sequences goes to zero. Then $S \subset X \subset \mathcal{H}$ is still orthonormal in $\mathcal{H}$, so it follows (from what fact?) that $\mathcal{H}$ is **not** a separable Hilbert space!

Remark on 7. Either (a) or (c) show:

$$r < p < s \quad \Longrightarrow \quad L^r(A) \cap L^s(A) \subset L^p(A).$$