ASSIGNMENT #10

(*All Problems Due* **Wednesday 12/5/01**.)

Additional XIV. Suppose $\langle f_n \rangle \subset C([a,b])$ is a sequence which converges in the $L^\infty([a,b])$ norm, that is, there exists $f \in L^\infty$ such that $\|f - f_n\|_\infty \to 0$.

Prove that in fact f is continuous, that is, that there exists a continuous representative of the equivalence class $[f] \in L^\infty$.

Remark. Suppose $(X, \|\cdot\|)$ is a Banach space. We call $A \subset X$ a *Banach subspace* of X if A is a Banach space in the norm $\|\cdot\|$. (That is, one needs to check that A is a vector subspace of X and that A is complete in the norm on X. But checking this completeness only requires checking that A is closed, that is, that if $\langle f_n \rangle$ is a sequence from A that has a limit in X, then in fact the limit is in A.) The above exercise shows $C([a,b]) \subset L^\infty([a,b])$ is a Banach subspace.

Additional XV. Consider $L^1([0,1])$ and fix $a \in [0,1]$.

(i) Prove that $F_1(f) = f(a)$ is *not* a bounded linear functional on $L^1([0,1])$.

(ii) Fix $\delta > 0$. Prove that

$$F_2(f) = \frac{1}{2\delta} \int_{a-\delta}^{a+\delta} f(x)dx$$

is a bounded linear functional on $L^1([0,1])$.

(iii) Prove that

$$F_3(f) = \lim_{\delta \to 0^+} \frac{1}{2\delta} \int_{a-\delta}^{a+\delta} f(x)dx$$

is *not* a bounded linear functional on $L^1([0,1])$.

For the next exercise you will need the following:

Theorem. If $f : [0,1] \to \mathbf{R}$ is measurable and is differentiable for almost every $x \in [0,1]$ and if $f' \in L^1([0,1])$ then

$$\int_0^x f'(t)\,dt = f(x) - f(0).$$

(The *proof* is to use theorems 10 and 14 in Chapter 5).

Additional XVI. Show that if $f : [0,1] \to \mathbf{R}$ is measurable and is differentiable for almost every $x \in [0,1]$ and if $f' \in L^1([0,1])$ then $f \in L^1([0,1])$.

[*Hint*: See 4.3 exercise #5!]

Additional XVII. (i) Let

$$W_1^1([0,1]) = \Big\{ f : [0,1] \to \mathbf{R} \Big| f \text{ is measurable and differentiable for} \\ \text{almost every } x \in [0,1] \text{ and } f' \in L^1([0,1]) \Big\}.$$

Define

$$\|f\| = \int_0^1 |f| + \int_0^1 |f'| = \|f\|_1 + \|f'\|_1$$

for $f \in W_1^1([0,1])$. Show $W_1^1([0,1])$ is a normed vector space with this norm. Show that it is complete (that is, a Banach space). [*Hint*: No need to go through a "from scratch" completeness proof. Use what you know ...]

(ii) Fix $a \in [0,1]$. Show

$$F(f) = f(a)$$

is a bounded linear functional on $W_1^1([0,1])$. [*Hint*: Use the theorem given before the previous exercise.]

Remark. $W_1^1([0,1])$ is called a *Sobolev space*. Sobolev invented these spaces to fix the problem in **XV**(i). In solving partial differential equations one can use the L^p spaces, but to get classical solutions, with actual defined values at points, required a bit more ...

Section 6.5, # 21.

Section 6.5, # 22.