Finite-dimensional spectral theory

part II: understanding the spectrum (and singular values)

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MATH 617 Functional Analysis

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Outline

what happened in part I

- see part I first: bueler.github.io/M617S20/slides1.pdf
- *definition.* for a square matrix $A \in \mathbb{C}^{n\times n}$, the *spectrum* is the set

$$\sigma(A) = \left\{\lambda \in \mathbb{C} \,\middle|\, Av = \lambda v \text{ for some } v \neq 0\right\}$$

- we proved:

  $A = QTQ^*$ *Schur decomposition* for any $A \in \mathbb{C}^{n\times n}$
  $A = Q\Lambda Q^*$ *spectral theorem* for normal ($AA^* = A^*A$) matrices

  where Q is unitary, T is upper-triangular, and Λ is diagonal
  - both decompositions "reveal" the spectrum:

$$\sigma(A) = \{\text{diagonal entries of } T \text{ or } \Lambda\}$$

  - spectral theorem for hermitian matrices is sometimes called the *principal axis decomposition* for quadratic forms

goal for MATH 617

goal

extend the spectral theorem to ∞-dimensions

- only possible for linear operators on Hilbert spaces H
  - inner product needed for adjoints and unitaries
  - unitary maps needed because they preserve vector space *and* metric *and* adjoint structures
- textbook (Muscat) extends to compact normal operators on H
  - the spectrum is eigenvalues (almost exclusively)
- recommended text (B. Hall, *Quantum Theory for Mathematicians*) extends further to bounded (continuous) normal operators on H
  - spectrum is not only eigenvalues
  - statement of theorem uses projector-valued measures
- Hall also extends to unbounded normal operators on H
  - but we won't get there ...
- the Schur decomposition has no straightforward extension

important class: unitary matrices

- back to matrices!

Definition

$U \in \mathbb{C}^{n\times n}$ is *unitary* if $U^* U = I$

Lemma

Consider $\mathbb{C}^n$ as a inner product space with $\langle v, w\rangle = v^* w$ and $\|v\|_2 = \sqrt{\langle v, v\rangle}$. Suppose U is linear map on $\mathbb{C}^n$. The following are equivalent:

- *U is unitary*
- *expressed in the standard basis, the columns of U are ON basis of $\mathbb{C}^n$*
- *$\langle Uv, Uw\rangle = \langle v, w\rangle$ for all $v \in \mathbb{C}^n$*
- *$\|Uv\|_2 = \|v\|_2$ for all $v \in \mathbb{C}^n$*
- *U is a metric-space isometry*

important class: normal matrices

Definition

$A \in \mathbb{C}^{n \times n}$ is *normal* if $A^* A = AA^*$

- includes: hermitian ($A^* = A$), unitary, skew-hermitian ($A^* = -A$)

Lemma

Consider $\mathbb{C}^n$ as a inner product space with $\langle v, w\rangle = v^* w$ and $\|v\|_2 = \sqrt{\langle v, v\rangle}$. Suppose A is linear map on $\mathbb{C}^n$. The following are equivalent:

- *A is normal*
- *$\|Ax\|_2 = \|A^* x\|_2$ for all x*
- *exists an ON basis of eigenvectors of A*
- *exists Q unitary and Λ diagonal so that $A = Q\Lambda Q^*$ (spectral theorem)*

Outline

power series of matrices

- suppose A is diagonalizable: $A = S\Lambda S^{-1}$
  - where S is invertible and Λ is diagonal
  - diagonal entries of Λ are eigenvalues of A
  - if A is normal (e.g. hermitian) then choose $S = Q$ unitary so $S^{-1} = Q^*$
- powers of A:

$$A^k = S\Lambda S^{-1} S\Lambda S^{-1} S\Lambda S^{-1} \cdots S\Lambda S^{-1} = S\Lambda^k S^{-1}$$

- if $f(z)$ is a power series then we can create $f(A)$:

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \qquad \Longrightarrow \qquad \begin{aligned} f(A) &= \sum_{n=0}^{\infty} c_n A^n = S\left(\sum_{n=0}^{\infty} c_n \Lambda^n\right) S^{-1} \\ &= S \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} S^{-1} \end{aligned}$$

- for example: $$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = S \begin{bmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{bmatrix} S^{-1}$$

what does "functional calculus" mean?

- given $A \in \mathbb{C}^{n\times n}$, a (finite-dimensional) *functional calculus* is algebraic-structure-preserving map from a set of functions $f(z)$ defined on $\mathbb{C}$ to matrices $f(A) \in \mathbb{C}^{n\times n}$
- example: for $f(z)$ analytic,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \qquad \Longrightarrow \qquad f(A) = \sum_{n=0}^{\infty} c_n (A - z_0 I)^n$$

$$= S \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} S^{-1}$$

- but ...
  - does the matrix power series $f(A) = \sum_{n=0}^{\infty} c_n (A - z_0 I)^n$ converge?
    reasonable question
  - does $f(z)$ have to be analytic anyway?
    no

norms of powers

- for any induced norm:
$$\|A^k\| \le \|A\|^k$$
- if A is diagonalizable then in any induced norm
$$\|A^k\| = \|S\Lambda^k S^{-1}\| \le \kappa(S) \max_{\lambda\in\sigma(A)} |\lambda|^k = \kappa(S)\rho(A)^k$$
    - $\kappa(S) = \|S\|\|S^{-1}\|$ is the *condition number* of S
    - $\rho(A) = \max_{\lambda\in\sigma(A)} |\lambda|$ is the *spectral radius* of A
    - $\rho(A) \le \|A\|$
- *corollary.* if A is diagonalizable and $\rho(A) < 1$ then $A^k \to 0$ as $k \to \infty$
    - actually this holds for all square A ... use the Schur or Jordan-canonical-form decompositions
- if A is normal then, because unitaries preserve 2-norm,
$$\|A^k\|_2 = \|Q\Lambda^k Q^*\|_2 = \max_{\lambda\in\sigma(A)} |\lambda|^k = \rho(A)^k$$
    - thus $\|A^k\|_2 = \|A\|_2^k$
    - note $\kappa_2(Q) = 1$ for a unitary matrix Q

does it converge?

$$f(A) \overset{*}{=} \sum_{n=0}^{\infty} c_n (A - z_0 I)^n$$

Lemma

Suppose $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ has radius of convergence $R > 0$. If $\|A - z_0 I\| < R$ in some induced norm then sum $*$ converges in that norm.

- if A is normal then $A = Q\Lambda Q^*$ so

$$\|A - z_0 I\|_2 = \max_{\lambda \in \sigma(A)} |\lambda - z_0| = \rho(A - z_0 I)$$

- in general $\rho(A - z_0 I) \leq \|A - z_0 I\|$ can be strict inequality

defining $f(z)$

- compare two ways of defining $f(A)$:

$$f(A) \overset{(1)}{=} \sum_{n=0}^{\infty} c_n (A - z_0 I)^n \qquad \text{and} \qquad f(A) \overset{(2)}{=} S \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} S^{-1}$$

- for (1) f needs to be analytic and have sufficiently-large radius of convergence relative to norm $\|A - z_0 I\|$
- for formula (2), A needs to be diagonalizable, but $f(z)$ does not need to be analytic ... it only needs to be defined on $\sigma(A)$

Theorem

If $A \in \mathbb{C}^{n\times n}$ is normal, if $\sigma(A) \subseteq \Omega \subseteq \mathbb{C}$, and if $f : \Omega \to \mathbb{C}$, then there is a unique matrix $f(A) \in \mathbb{C}^n$ so that:

1. *$f(A)$ is normal*
2. *$f(A)$ commutes with A*
3. *if $Av = \lambda v$ then $f(A)v = f(\lambda)v$*
4. $\|f(A)\|_2 = \max_{\lambda\in\sigma(A)} |f(\lambda)|$

proof. By the spectral theorem there is a unitary matrix Q and a diagonal matrix Λ so that $A = Q\Lambda A^*$, with columns of Q which are eigenvectors of A and all eigenvalues of A listed on the diagonal of Λ. Define

$$f(A) = Q \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} Q^*.$$

It has the stated properties. It is a unique because its action on a basis (eigenvectors of A) is determined by property 3.

the meaning of the functional calculus

- if A is normal then you can apply any function $f(z)$ to it, giving $f(A)$, as though A is "just like a complex number"
  - f merely has to be defined[1] on the finite set $\sigma(A)$
  - the matrix 2-norm behaves well: $\|f(A)\|_2 = \max_{\lambda\in\sigma(A)} |f(\lambda)|$
  - eigendecomposition is therefore powerful when A is normal!
- if A is diagonalizable then $f(A)$ can be *defined* the same:

$$f(A) = S \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} S^{-1}$$

  but surprising behavior is possible: $\|f(A)\| \gg \max_{\lambda\in\sigma(A)} |f(\lambda)|$
- if A is defective then what? revert to using power series just to define $f(A)$?

[1] In ∞-dimensions f needs some regularity. Thus there are separate wikipedia pages on *holomorphic functional calculus*, *continuous functional calculus*, and *borel functional calculus*.

functional calculus applications

1. suppose A is hermitian and we want to build a unitary matrix from it
   - A is normal and $\sigma(A) \subset \mathbb{R}$

solution 1. $f(z) = e^{iz}$ maps $\mathbb{R}$ to the unit circle so

$$U = e^{iA} \quad \text{is unitary}$$

solution 2. $f(z) = \dfrac{z+i}{z-i}$ maps $\mathbb{R}$ to the unit circle so

$$U = (A + iI)(A - iI)^{-1} \quad \text{is unitary}$$

2. suppose U is unitary and we want to build a hermitian matrix from it
   - U is normal and $\sigma(U) \subset S^1 = \{z \in \mathbb{C} : |z| = 1\}$

solution. $f(z) = \mathrm{Log}(z)$ maps the unit circle S^1 to the real line, so

$$A = \frac{1}{i}\,\mathrm{Log}(U) = -i\,\mathrm{Log}(U) \quad \text{is hermitian}$$

functional calculus applications: linear ODEs

3. given $A \in \mathbb{C}^{n\times n}$ normal, and given $y_0 \in \mathbb{C}$, solve

$$\frac{dy}{dt} = Ay, \qquad y(t_0) = y_0$$

for $y(t) \in \mathbb{C}^n$ on $t \in [t_0, t_f]$

solution. $y(t) = e^{tz}$ solves $dy/dt = zy$ so, using the functional calculus with $f(z) = e^{(t-t_0)z}$,

$$\begin{aligned} y(t) &= e^{(t-t_0)A} y_0 \\ &= \texttt{expm((t-t0)*A)*y0}, \\ \|y(t)\|_2 &= e^{(t-t_0)\omega(A)} \|y_0\|_2 \end{aligned}$$

where $\omega(A) = \max_{\lambda\in\sigma(A)} \operatorname{Re} \lambda$

- if A is diagonalizable $A = S\Lambda S^{-1}$ then the same applies ... except the norm of the solution includes $\kappa(S)$
- if A is defective then the general solution of the ODE system is *not* exponential

4. ∞-dimensional version: Schrödinger's equation in quantum mechanics

Outline

Definition

given $A \in \mathbb{C}^{n\times n}$ then $\mathbb{C} \setminus \sigma(A)$ is the *resolvent set*, and if $z \in \mathbb{C} \setminus \sigma(A)$ then

$$R_z(A) = (A - zI)^{-1}$$

is the *resolvent matrix*

- recall: $z \in \sigma(A)$ if and only if $A - zI$ is not invertible
- the resolvent set $\mathbb{C} \setminus \sigma(A)$ is open
- $R_0(A) = A^{-1}$ if $0 \notin \sigma(A)$
- $R_z(A)$ "resolves" the equation $Av - zv = b$

- if $A = S\Lambda S^{-1}$ is diagonalizable and $z \in \mathbb{C} \setminus \sigma(A)$ then

$$R_z(A) = \left(S\Lambda S^{-1} - zSIS^{-1}\right)^{-1} = S\,(\Lambda - zI)^{-1}\,S^{-1}$$

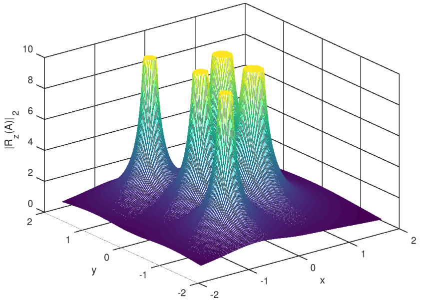
so in any induced norm

$$\|R_z(A)\| \le \|S\|\|S^{-1}\|\|\,(\Lambda - zI)^{-1}\,\| = \kappa(S) \max_{\lambda\in\sigma(A)} |\lambda - z|^{-1}$$

- if A is normal then we can choose $S = Q$ unitary with $\kappa_2(Q) = 1$ so

$$\|R_z(A)\|_2 = \max_{\lambda\in\sigma(A)} |\lambda - z|^{-1}$$

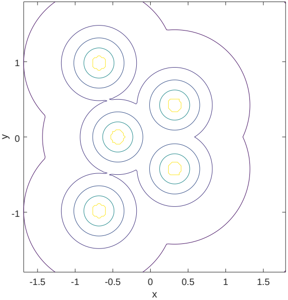
- one may plot $g(z) = \|R_z(A)\|$

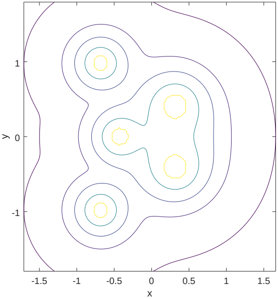


resolvent norms illustrated

- contours of $z \mapsto \|R_z(A)\|_2 = \|(A - zI)^{-1}\|_2$ is best spectral picture?

```
>> [A,B] = gennormal(5);   % A,B have same eigs; A normal but B not
>> resolveshow(A)          % normal case      (LEFT)
>> resolveshow(B)          % nonnormal case   (RIGHT)
```

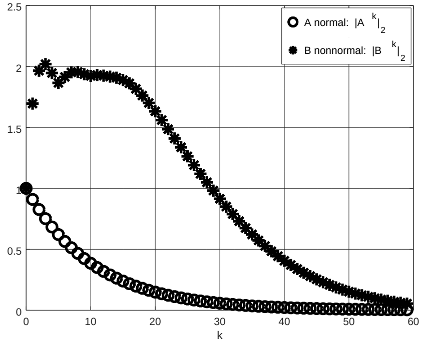




- last slide already proved contours would be round for normal A
- $\sigma_\epsilon(A) = \left\{z \in \mathbb{C} \,:\, \|(A - zI)^{-1}\|_2 \geq \epsilon^{-1}\right\}$ is the ϵ-*pseudospectrum* of A

- facts and definitions:
  - $\|A^k\| \leq \|A\|^k$ in any induced norm
  - $\rho(A) = \max_{\lambda\in\sigma(A)} |\lambda|$
  - if A is normal then $\|A^k\|_2 = (\|A\|_2)^k = \rho(A)^k$
  - if $\rho(A) < 1$ then $A^k \to 0$ as $k \to \infty$ *proof?*
- but if A is not normal and $\rho(A) < 1$ then $\|A^k\|_2$ *can be big* for a while
  - e.g. random 100×100 matrices A,B with $\rho(A) = \rho(B) < 1$

```
>> max(abs(eig(A)))
ans =  0.90909
>> max(abs(eig(B)))
ans =  0.90909
```



redefining “spectrum”: nonexistence of resolvent

Definition

given $A \in \mathbb{C}^{n\times n}$, the *spectrum of A* is the set

$$\sigma(A) = \left\{\lambda \in \mathbb{C} \,\middle|\, A - \lambda I \text{ does not have a bounded inverse}\right\}$$

- in $\mathbb{C}^n$ this is the same as our original definition:

$$\sigma(A) = \left\{\lambda \in \mathbb{C} \,\middle|\, Av = \lambda v \text{ for some } v \neq 0\right\}$$

- in ∞-dimensions it is different because there exist one-to-one bounded operators which do not have bounded inverses
  - *example 1*: the one-to-one right-shift operator R on ℓ^1 has spectrum[2] $\sigma(R) = \{z \in \mathbb{C} : |z| \leq 1\}$, but it has no eigenvalues
  - *example 2*: the hermitian multiplication operator $(Mf)(x) = xf(x)$ on $L^2[0,1]$ has no eigenvalues but $\sigma(M) = [0,1]$

[2] we will prove this by showing that $\sigma(L) = \{z \in \mathbb{C} : |z| \leq 1\}$ for the left-shift operator $L = R^*$, based on eigenvalues, and that $\sigma(A^*) = \sigma(A)$ in a Banach algebra

Outline

Definition

$P \in \mathbb{C}^{n\times n}$ is an *orthogonal projector* if $P^2 = P$ and $P^* = P$

- as for any projector ($P^2 = P$):

$$\ker P = \operatorname{im}(I - P), \quad \operatorname{im} P = \ker(I - P), \quad \mathbb{C}^n = \ker P \oplus \operatorname{im} P, \quad \sigma(P) \subset \{0, 1\}$$

- but for orthogonal projectors:

$$\ker P \perp \operatorname{im} P$$

  - *proof.* if $u \in \ker P$ and $v = Pz \in \operatorname{im} P$ then $u^* v = u^*(Pz) = (Pu)^* z = 0$
- orthogonal projectors are hermitian, thus normal
- examples:

$$0, \quad I, \quad P = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$$

constructing orthogonal projectors from ON vectors

- since P is hermitian and $\sigma(P) \subset \{0, 1\}$, the spectral theorem plus re-ordering of the columns of Q gives

$$P = Q\Lambda Q^* = Q \begin{bmatrix} \hat{I} & \\ & 0 \end{bmatrix} Q^* = \hat{Q}\hat{Q}^*$$

where $\hat{I}$ is a $k \times k$ identity and $\hat{Q}$ is a $n \times k$ matrix of columns of Q

Lemma

$P \in \mathbb{C}^{n\times n}$ is an orthogonal projector if and only if there exist ON vectors $q_1, \dots, q_k$, for $0 \le k \le n$, so that

$$P = \hat{Q}\hat{Q}^* \quad \text{and} \quad \hat{Q} = \left[\begin{array}{c|c|c|c} q_1 & q_2 & \dots & q_k \end{array}\right] \in \mathbb{C}^{n\times k}$$

- hard direction of proof is above; easy direction: $(\hat{Q}\hat{Q}^*)^2 = \dots$
- note $\hat{Q}^*\hat{Q} = \hat{I}$
- rank 1 case: $P = qq^* = (aa^*)/(a^*a)$
- construction from full-column-rank A: $P = A(A^*A)^{-1}A^*$

spectral theorem = decomposition into projectors

- consider this calculation for A normal:

$$A = Q\Lambda Q^* = Q\left(\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}\right) Q^*$$

$$= Q\left(\begin{bmatrix} \lambda_1 & & \\ & & \\ & & \end{bmatrix} + \dots + \begin{bmatrix} & & \\ & & \\ & & \lambda_n \end{bmatrix}\right) Q^* = q_1\lambda_1 q_1^* + \dots + q_n\lambda_n q_n^*$$

$$= \sum_{j=1}^{n} \lambda_j q_j q_j^*$$

  - A decomposes into a linear combination of rank-one orthogonal projectors

- thus normal matrices act on vectors like this:

$$Av = \sum_{j=1}^{n} \lambda_j q_j q_j^* v = \sum_{j=1}^{n} \lambda_j \langle q_j, v\rangle q_j$$

  - this formula appears in most applications of normal operators

resolution of the identity

- if A is normal then $A = \sum_{i=1}^{n} \lambda_i q_i q_i^*$ where $\{q_i\}$ are ON
- if A is normal then we can use its eigenvectors to decompose the identity:

$$I = QQ^* = \sum_{i=1}^{n} q_i q_i^*$$

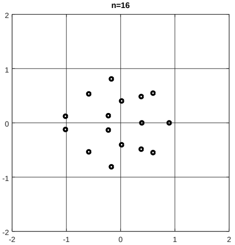
  - called a *resolution of the identity*
- application: Parseval's identity for any ON basis

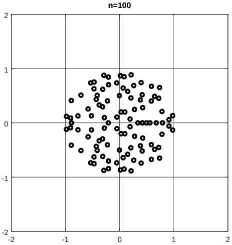
$$\|v\|_2^2 = v^* v = v^* I v = \sum_{i=1}^{n} v^* q_i q_i^* v = \sum_{i=1}^{n} |\langle q_i, v\rangle|^2$$

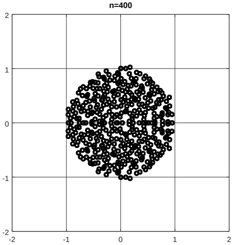
spectra of big random matrices

- *claim* (*circular law*). if $A \in \mathbb{R}^{n\times n}$ has entries which are normally-distributed random variables with mean zero and variance n^{-1}, so $a_{ij} \sim N(0, n^{-1})$, then as $n \to \infty$ the spectrum of A fills the unit disc

```
>> A = randn(n,n)/sqrt(n);
>> lam = eig(A);
>> plot(real(lam),imag(lam),'o'),  grid on,  axis([-2 2 -2 2])
```







- but these matrices are not normal

spectra of big random *normal* matrices

- but `randn(n,n)` is not normal (i.e. normal with probablility zero)
- construct a random *normal* matrix with the same spectrum:

```
function [A,B] = gennormal(n);
% GENNORMAL  Generate a random n x n complex matrix A which is normal
% (but not hermitian).  The entries have normal distributions.  The
% eigenvalues will roughly cover the unit disc when n is large.  Also
% returns B, a nonnormal matrix with the same eigenvalues as A.
% Example:
%   >> [A,B] = gennormal(100);
%   >> lam = eig(A);
%   >> plot(real(lam),imag(lam),'o'),  grid on    % same picture for B
%   >> norm(A'*A - A*A')      % very small
%   >> norm(B'*B - B*B')      % not small
% See also GENHERM, PROJMEASURE.

B = randn(n,n)/sqrt(n);   % https://en.wikipedia.org/wiki/Circular_law
                          %     says eigenvalues of B are asymptotically
                          %     uniformly distributed on unit disc
[X,D] = eig(B);           % D is diagonal and holds eigenvalues and
                          %     X holds (nonorthogonal) eigenvectors
[Q,R] = qr(X);            % Q holds ON basis for C^n, built from applying
                          %     orthogonalization to columns of X
A = Q*D*Q';               % construct A to be normal but to have same
                          %     eigenvalues as B
```

spectral subsets correspond to orthogonal projectors

- I also wrote a code `projmeasure.m` which shows $\sigma(A)$ as a subset of $\mathbb{C}$ and lets you select the eigenvalues for which you want eigenvectors
- demo 1:

```
>> A = gennormal(100);
>> P = projmeasure(A);    % <-- user input with mouse
                          %     selects a projector
>> k = rank(P)            % = number of selected eigenvalues
```

- demo 2:

```
>> A = expm(i*eye(6) + gennormal(6));
>> [P,Qh] = projmeasure(A);
>> Qh                     % view selected eigenvectors
```

- demo 3:

```
>> U = expm(i*genherm(10));   % random unitary matrix
>> [P,Qh] = projmeasure(U);
>> Qh                         % view selected eigenvectors
```

projector-valued measures (von Neumann)

- John von Neumann imagined these kind of spectral pictures in the 1920s
  - before he invented electronic computers in the 1940s
- he proposed a *projector-valued measure* E_λ for each $A \in B(\mathbb{C}^n)$ normal
  - if $Z \subset \sigma(A) \subset \mathbb{C}$ then $P = E_\lambda(Z)$ is an orthogonal projector
  - $\operatorname{im} P = \operatorname{im} E_\lambda(Z)$ is span of eigenvectors for eigenvalues $\lambda \in Z$
- he built this to handle quantum mechanical operators rigorously
- *(von Neumann's) spectral theorem.* if $A \in B(H)$ normal, for H a Hilbert space, then there exists a projector-valued measure E_λ so that

$$A = \int_{\sigma(A)} \lambda \, dE_\lambda$$

- the most general functional calculus follows immediately:

$$f(A) = \int_{\sigma(A)} f(\lambda) \, dE_\lambda$$

  - f is merely measurable
  - A could even be unbounded (i.e. not Lipschitz)

Outline

why singular values?

- eigenvalues can be useful!
- but they are only defined for square matrices
  - in ∞-dimensions: "spectrum is useful, but only for $B(X)$, not $B(X, Y)$"
- ... and sometimes not so useful anyway
  - only "safe" to use eigenvalues if eigenvectors are orthogonal (A normal)
  - diagonalization $A = S\Lambda S^{-1}$ may tell us little about A when $\kappa(S) \gg 1$
  - square matrices can be defective anyway
- however, *any* $A \in \mathbb{C}^{m\times n}$ has *singular values*
  - what do the eigenvalues say?

    Behavior of powers A^k or functions $f(A)$ like e^{At}.
  - what do the singular values say?

    Invertibility of A: rank, nullity

    Geometric action of A: $\|A\|_2$, $\|A^{-1}\|_2$, condition number, ϵ-pseudospectrum
  - so, what information do you want?

visualizing a matrix

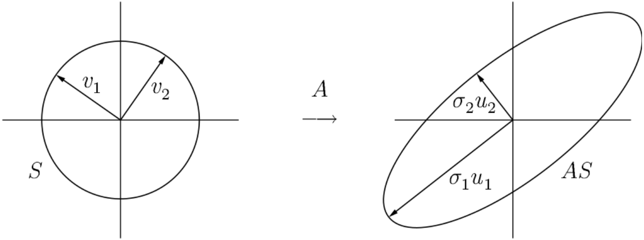


Figure 4.1. *SVD of a* 2×2 *matrix.*

figure from Trefethen & Bau, Numerical Linear Algebra, SIAM Press 1997

- $A \in \mathbb{R}^{m \times n}$ sends the unit sphere in $\mathbb{R}^n$ to a possibly-degenerate hyperellipsoid in $\mathbb{R}^m$
  - this is the fundamental way to visualize a linear operator!
  - also true for $A \in \mathbb{C}^{m \times n}$ . . . but less visualizable
- the *singular values* of A define the geometry of the output hyperellipsoid

Theorem

if $A \in \mathbb{C}^{m\times n}$ then there exist $U \in \mathbb{C}^{m\times m}$ unitary, $V \in \mathbb{C}^{n\times n}$ unitary, and $\Sigma \in \mathbb{R}^{m\times n}$ diagonal, with nonnegative entries, so that

$$A = U\Sigma V^*$$

- *singular value decomposition (SVD)* of A
- diagonal entries σ_i of Σ are the *singular values* of A
  - note Σ is same shape as A, while U, V are always square
  - normalization $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min\{m,n\}}$ makes Σ unique
  - if $A \neq 0$ then $\sigma_1 > 0$
  - if $A = 0$ we take $\Sigma = 0$ and choose U, V as any unitaries
- action of $A = U\Sigma V^*$ on a vector:
  - multiplication by V^* finds coefficients of the vector in the columns of V
  - multiplication by Σ stretches the vector along standard axes
  - multiplication by U rotates the vector to the output hyperellipsoid

singular value decomposition: examples

- *example 1.* if $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ then

$$A = \begin{bmatrix} -0.92388 & -0.38268 \\ -0.38268 & 0.92388 \end{bmatrix} \begin{bmatrix} 5.3983 & \\ & 0.92621 \end{bmatrix} \begin{bmatrix} -0.75545 & -0.6552 \\ -0.6552 & 0.75545 \end{bmatrix}^*$$

  - $\|A\|_2 = 5.3983$, $\|A^{-1}\|_2 = 1/0.92621$
  - compare: $\sigma(A) = \{5, 1\}$

- *example 2.* if $B = \begin{bmatrix} 6 & 5 \\ 4 & 3 \\ 1 & 2 \end{bmatrix}$ then

$$B = \begin{bmatrix} -0.82264 & -0.05242 & -0.56614 \\ -0.52578 & -0.30878 & 0.79259 \\ -0.21636 & 0.94969 & 0.22646 \end{bmatrix} \begin{bmatrix} 9.49393 & \\ & 0.93025 \\ & \end{bmatrix} \begin{bmatrix} -0.76421 & -0.64497 \\ -0.64497 & 0.76421 \end{bmatrix}^*$$

  - $\|B\|_2 = 9.49393$
  - B is not invertible
  - $\sigma(B)$ is not defined

singular value decomposition: proof

proof. Induct on n, the column size of A. If $n=1$ then $A=[a]$ where $a\in\mathbb{C}^m$. Then

$$U=\left[\frac{a}{\|a\|_2}\right],\quad \Sigma=[\|a\|_2],\quad V=[1]$$

is an SVD for A.

For $n>1$ let $v_1\in\mathbb{C}^n$ be a unit vector which maximizes the continuous function

$$f(x)=\|Ax\|_2$$

over the compact set $S^n=\{x\in\mathbb{C}^n\,:\,\|x\|_2=1\}$. (*We just used finite-dimensionality!*) Then Av_1 is a vector in $\mathbb{C}^m$ with length $\sigma_1=\|Av_1\|_2=\|A\|_2$. If $\sigma_1=0$ we are done because A is the zero matrix. (*Why?*) Otherwise $\sigma_1>0$ so let $u_1=Av_1/\sigma_1$. Now we have $Av_1=\sigma_1 u_1$.

Extend v_1 and u_1 to orthonormal bases of $\mathbb{C}^n,\mathbb{C}^m$, respectively, giving unitary matrices

$$\tilde V=\left[\begin{array}{c|c|c|c} v_1 & \tilde v_2 & \dots & \tilde v_n\end{array}\right],\qquad \tilde U=\left[\begin{array}{c|c|c|c} u_1 & \tilde u_2 & \dots & \tilde u_m\end{array}\right].$$

Now apply A to $\tilde V$,

$$A\tilde V=\left[\begin{array}{c|c|c|c} \sigma_1 u_1 & w_2 & \dots & w_n\end{array}\right].$$

Next apply $\tilde U^*$, and note that $\tilde U^* u_1=e_1$:

$$\tilde U^* A\tilde V=\left[\begin{array}{c|c} \sigma_1 & z^* \\ \hline 0 & M\end{array}\right]$$

singular value decomposition: proof cont.

cont. We have

$$\tilde{U}^* A \tilde{V} = \left[\begin{array}{c|c} \sigma_1 & z^* \\ \hline 0 & M \end{array}\right]$$

for $z \in \mathbb{C}^{n-1}$ and $M \in \mathbb{C}^{(m-1)\times(n-1)}$. Because $\tilde{U}$, $\tilde{V}$ are unitary, the matrix norm is unchanged: $\|\tilde{U}^* A \tilde{V}\|_2 = \|A\|_2$.

In fact $z = 0$, for the following reason. Let $w \in \mathbb{C}^m$ be the vector $w = \begin{bmatrix} \sigma_1 \\ z \end{bmatrix}$. It is nonzero because $\|w\|_2 = (\sigma_1^2 + \|z\|_2^2)^{1/2} \ge \sigma_1 > 0$. But

$$\left\| \left[\begin{array}{c|c} \sigma_1 & z^* \\ \hline 0 & M \end{array}\right] \begin{bmatrix} \sigma_1 \\ z \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} \sigma_1^2 + z^* z \\ Mz \end{bmatrix} \right\|_2 \ge \sigma_1^2 + \|z\|_2^2 = (\sigma_1^2 + \|z\|_2^2)^{1/2} \|w\|_2.$$

That is, $\|\tilde{U}^* A \tilde{V} w\|_2 \ge (\sigma_1^2 + \|z\|_2^2)^{1/2} \|w\|_2$, so if $z \ne 0$ then $\|A\|_2 = \|\tilde{U}^* A \tilde{V}\|_2 > \sigma_1$, contradicting the definition of σ_1.

Thus

$$\tilde{U}^* A \tilde{V} = \left[\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & M \end{array}\right]$$

By the induction hypothesis there exist $\hat{U}, \hat{\Sigma}, \hat{V}$ so that $M = \hat{U}\hat{\Sigma}\hat{V}^*$. Since products of unitaries are unitary, we have an SVD of A:

$$A = \left(\tilde{U} \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \hat{U} \end{array}\right]\right) \left[\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & \hat{\Sigma} \end{array}\right] \left(\tilde{V} \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \hat{V} \end{array}\right]\right)^* = U \Sigma V^* \qquad \square$$

- $\|A\|_2 = \|\Sigma\|_2 = \sigma_1$
- α is a singular value of A if and only if α^2 is an eigenvalue of $A^* A$
- the singular values of A are the same as those of A^*
- for any $A \in \mathbb{C}^{m\times n}$,
  - $\text{rank}(A) = k$ where $\sigma_k > 0$ and $\sigma_{k+1} = 0$
  - $\text{nullity}(A) = q$ where q is number of zero singular values ($m \ge n$)
- if $A \in \mathbb{C}^{n\times n}$ is square then
  - $|\det(A)| = \prod_{j=1}^{n} \sigma_j$
  - if A is invertible then $\|A^{-1}\|_2 = 1/\sigma_n$
  - $\kappa_2(A) = \sigma_1/\sigma_n \in [1, \infty]$ is the eccentricity of the output hyperellipsoid
  - $\sigma_n \le \min_{\lambda\in\sigma(A)} |\lambda| \le \max_{\lambda\in\sigma(A)} |\lambda| \le \sigma_1$
- if A is square and normal then $\sigma_j = |\lambda_j|$ (with ordering of $\sigma(A)$)

Outline

please try reading the textbook backwards now

- go to the end of Chapter 15 "C^* algebras" and read backwards:
    - von Neumann's spectral theorem for bounded operators on Hilbert spaces
    - functional calculus for normal elements
    - singular value decomposition for compact operators between Hilbert spaces
    - spectral theorem for compact normal operators on a Hilbert space
    - definition of *normal*, *unitary*, and *self-adjoint* (hermitian) elements
    - definition of a C^* algebra
- on the other hand, go to the beginning of Chapter 14 "Spectral theory" and read forward
- I hope that by the end of the semester it will make sense!