corrected Assignment #4

Due Monday, 17 February 2020, at the start of class

At this point you should have read all of Chapters 7 and 8 of the textbook, and be comfortable with this material. You should also be comfortable with the content of the slides *Finite-dimensional spectral theory I* at

bueler.github.io/M617S20/slides1.pdf

Note that section 7.5 covers infinite series, but why is it there? If you look ahead to the end of Chapter 12 you see more complex analysis, namely Taylor's and Cauchy's theorems about analytic functions $f : \mathbb{C} \to \mathbb{C}$. Then, looking into Chapter 13 on Banach algebras, you will see that series and analytic functions *are applied to operators*. Exercises **C**, **D**, **E** preview this "Banach algebras" material, but in the finite-dimensional case.

One exercise below is identified with your initials. Please LaTeX this problem and send both the `.tex` and `.pdf` to me at `elbueler@alaska.edu` by the due date.

DO THE FOLLOWING EXERCISES from the textbook (Muscat, *Functional Analysis*, 2014):

- #13 in Exercises 8.10, page 126. $\longleftarrow$ **OS**
- #17 in Exercises 8.10, page 127.
- #2 in Exercises 8.14, page 129. $\longleftarrow$ **DD**
- #3 in Exercises 8.14, page 129. $\longleftarrow$ **WV**
- #2 in Exercises 8.17, page 131.

ALSO DO THE FOLLOWING EXERCISES.

- **Exercise C.** Consider the set of all $n \times n$ matrices with complex entries, denoted as $B(\mathbb{C}^n)$. (*From section 8.1, we know that $n \times n$ matrices are continuous linear maps on $\mathbb{C}^n$.*) Now, $B(\mathbb{C}^n)$ is a vector space because matrices can be added and multiplied by scalars. Furthermore, given a norm $\|\cdot\|$ on $\mathbb{C}^n$, for $A \in B(\mathbb{C}^n)$ we define

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|},$$

the *induced norm*. Theorem 8.7 shows $(B(\mathbb{C}^n), \|\cdot\|)$ is a normed vector space. Multiplication of square matrices is well-defined, as composition of linear maps, and Theorem 8.9 shows that $\|AB\| \le \|A\|\|B\|$. Thus $B(\mathbb{C}^n)$ is a *Banach algebra*, as defined on the first page of Chapter 13.

(a) Show that $\|I\| = 1$.

(b) Define the *spectral radius* of $A \in B(\mathbb{C}^n)$ as $\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$. Show that $\rho(A) \le \|A\|$.

- **Exercise D.** Consider $f(z) = 1/z$, a function which is analytic (i.e. complex-differentiable) on the punctured plane $\mathbb{C} \setminus \{0\}$. Given nonzero z_0, compute the Taylor series of f around the basepoint z_0 and find the radius of convergence R. As a particular case, write down this power series when $z_0 = 1$, and its R.
- **Exercise E.** Show that if $A \in B(\mathbb{C}^n)$, and if $\|I - A\| < 1$ in some induced matrix norm, then A is invertible and

$$A^{-1} = I + (I - A) + (I - A)^2 + \dots$$

(*Convergence is in the induced norm. Observe that you can approximately invert such matrices by applying a polynomial to them!*)