Assignment #9

Due Monday, 24 April at the start of class

Please read Chapters 9 and 10 of LeVeque. This Assignment emphasizes 9.1–9.6.

P33. Consider the following method, which applies centered differences to both sides of the equation, for solving the heat equation $u_t = u_{xx}$:

$$U_i^{n+2} = U_i^n + \frac{2k}{h^2}(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}).$$

This is called the *Richardson* method. (*L. F. Richardson did many things more important and successful than inventing this scheme.*)

a) Determine the order of accuracy of this method, in space and time. The answer will be in form $\tau(x,t) = O(k^p + h^q)$; determine p, q.

b) Derive the method by applying the midpoint ODE method, equation (5.23), to the MOL ODE system (9.10). By looking these things up in the textbook—give specific references—state the eigenvalues of A in (9.10) assuming Dirichlet boundary conditions at $x = 0$ and $x = 1$. Similarly, look up the region of absolute stability of the midpoint method (5.23).

c) What do you conclude? Is the method likely to generate reasonable results? Why or why not?

P34. Consider the heat equation $u_t = \kappa\, u_{xx}$ for $\kappa > 0$, $x \in [0,1]$, and Dirichlet boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$. Suppose we have initial condition $u(x,0) = \sin(5\pi x)$.

a) Confirm that

$$u(x,t) = e^{-25\pi^2 \kappa t} \sin(5\pi x)$$

is an exact solution to this problem. (*I claim it is* the *exact solution, because the problem is well-posed, but of course you do not have to show this.*)

b) Implement the backward Euler (BE) method, as applied to MOL ODE system (9.10), to solve this heat equation problem. Specifically, use diffusivity $\kappa = 1/20$ and final time $t_f = 0.1$. Note that you do not need to use Newton's method to solve the implicit equation, which is a linear system, but you should use sparse storage and MATLAB's backslash (or etc.).

c) What do you expect for the convergence rate $O(h^p)$? Then measure it by using the exact solution from **a)**, at the final time, and the infinity norm $\|\cdot\|_\infty$, and $h = 0.05, 0.02, 0.01, 0.005, 0.002, 0.001$. Make a log-log convergence plot of h versus the error, setting $k = 2h$ for the "refinement path".

d) Repeat parts **b)** and **c)** but with the trapezoidal rule instead of BE. (That is, implement and measure the convergence rate of Crank-Nicolson, with everything else the same.)

P35. Consider the Jacobi iteration (4.4) for the linear system $Au = f$ arising from a centered difference approximation of the boundary value problem $u_{xx}(x) = f(x)$. Show that this iteration can be interpreted as forward Euler time-stepping applied to the MOL equations like (9.10) and with time step $k = \frac{1}{2}h^2$. (I.e. the MOL equations are those arising from a centered difference discretization of the heat equation $u_t(x,t) = u_{xx}(x,t) - f(x)$.)

Comment. Note that if the boundary conditions are held constant then the solution of the time-dependent heat equation decays to the steady state solution (i.e. to the solution of $u_{xx} = f$ with those boundary values). However, while marching to steady state with an explicit method is one way to solve the steady state boundary value problem, it is a very inefficient way.

P36. Consider the following method for solving the advection equation $u_t + au_x = 0$, where a is constant

$$U_i^{n+1} = U_i^{n-1} - \frac{ak}{h}(U_{i-1}^n - U_{i+1}^n).$$

Again this applies centered differences to all derivatives. This is the *leapfrog* method.

a) Determine the order of accuracy of this method (in both space and time). The answer will be in form $\tau(x,t) = O(k^p + h^q)$; determine p, q.

b) State the MOL ODE system $U(t)' = AU(t)$ from which the above method comes. Assuming periodic boundary conditions on the interval $x \in [0,1]$, what are the eigenvalues of A? Then derive the method by applying the midpoint ODE method to it. By looking up the stability region of the midpoint method, explain what is understood about the stability of this PDE method. (*You may extract this from the book; give specific references.*)

c) Implement this leapfrog method on the following periodic boundary condition problem: $x \in [0,1]$, $a = 0.5$, $t_f = 10$, $u(x,0) = \sin(6\pi x)$. To make the implementation work you will have to compute the first step by some other scheme; describe and justify what you do.

d) What is the exact solution to the problem in part **c)**? Use $h = 0.1, 0.05, 0.02, 0.01, 0.005, 0.002$ and $k = h$ and show a log-log convergence plot using the infinity norm for the error. What $O(h^p)$ do you expect for the rate of convergence, and what do you measure?