Assignment #7

Due Friday, 31 March at the start of class

Please read Chapters 5 and 6 of LeVeque. Most of Chapter 5 is important, but I will mostly avoid section 5.9. In Chapter 6 I will not cover 6.4.

P26. **a)** In preparation for problem **P27** below, write two ODE solvers

```
function [tt,zz] = forwardeuler(f,eta,t0,tf,N)
function [tt,zz] = rk4(f,eta,t0,tf,N)
```

which implement schemes (5.19) and (5.33), respectively, to solve (5.1) and (5.2).

Here the inputs are `function z = f(u,t)`, a set of initial values `eta` $= u(t_0)$, the initial time `t0`, the final time `tf`, and the number of steps (subintervals) `N`. Note that the time step is $\Delta t = k = (t_f - t_0)/N$. The outputs are the entire trajectory. That is, `tt` is a 1D array of length $N+1$ starting with t_0 and ending with t_f. If $\eta \in \mathbb{R}^s$ then `zz` is a 2D array with s rows and $N+1$ columns; each column i gives the solution $u(t)$ at the ith time in `tt`.

b) Test these solvers on a simple problem with $s > 1$ for which you know an exact solution which is smooth.[1] Demonstrate that the global error converges at the expected rate as the timestep $k \to 0$. Of course, you must decide (and reference) what the expected rate is for each method. (*There is no need to compute local truncation errors yourself, but you must know their orders.*)

P27. (*This multi-part example is a real application. From this you will appreciate the our abstract notation for ODE systems, the vectorizability of* MATLAB *and related languages, and higher-order explicit ODE schemes! Rather famously, this problem has an exact solution, but you won't need it here.*)

Consider the problem of two massive bodies (particles) with masses m_1 and m_2. They are attracted by gravity only. They travel in a plane so their positions are given by vectors $\mathbf{x}_i(t) = (x_i(t), y_i(t))$ for $i = 1, 2$. Newton's second law and Newton's law of gravity combine to say:

(1)
$$\begin{aligned} m_1 \mathbf{x}_1'' &= -G m_1 m_2 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} \\ m_2 \mathbf{x}_2'' &= -G m_1 m_2 \frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_1 - \mathbf{x}_2|^3} \end{aligned}$$

where primes denote differentiation with respect to t, of course.

We will consider the Earth and the Moon in isolation as our example. Thus the constants are

$$\begin{aligned} m_1 &= 5.972 \times 10^{24}\,\text{kg}, \\ m_2 &= 7.348 \times 10^{22}\,\text{kg}, \\ G &= 6.674 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}, \end{aligned}$$

and we measure t in seconds and x_i, y_i in meters. (*Though it will not be graded, please confirm that the units balance in equations* (1).)

[1] For example, the problem $x'' + x = 0$, $x(0) = 1$, $x'(0) = 0$ has exact solution $x(t) = \cos t$, and solving it on $[t_0, t_f] = [0, 2]$, for instance, makes a reasonable test problem in this case.

a) By using notation $v_i = x_i'$, $w_i = y_i'$ for $i = 1, 2$, write system (1) as a first-order ODE system of dimension $s = 8$ with solution vector $u(t) \in \mathbb{R}^8$, a column vector, using the standard component ordering

$$u(t) = \begin{bmatrix} x_1(t) & y_1(t) & x_2(t) & y_2(t) & v_1(t) & w_1(t) & v_2(t) & w_2(t) \end{bmatrix}^\top$$
$$= \begin{bmatrix} u_1(t) & u_2(t) & u_3(t) & u_4(t) & u_5(t) & u_6(t) & u_7(t) & u_8(t) \end{bmatrix}^\top.$$

That is, write (1) in the form of (5.1) in the book: $u'(t) = f(u(t), t)$. (*In fact the right side of* this *ODE system does not have explicit dependence on t*.) Then implement a single MATLAB/etc. function

```
function z = fearthmoon(u,t)
```

which computes the right-hand-side function $f(u, t)$ of the ODE system. (*Avoid unnecessary memory usage in implementing this function. That is, do not unnecessarily duplicate values.*)

b) For initial conditions which are vaguely like what they are in reality, at least if you turned off all the gravity of other bodies, suppose $t_0 = 0$ and $x_1(0) = 0, y_1(0) = 0, v_1(0) = 0, w_1(0) = 0$ (i.e. start the Earth stationary at the origin) and[2] $x_2(0) = 3.844 \times 10^8$ meters, $y_2(0) = 0$, $v_2(0) = 0$, $w_2(0) = 1.022 \times 10^3\,\mathrm{m\,s^{-1}}$.

Use these initial conditions, and each of the solvers from problem **P26**, to generate four approximate solutions with $t_f = 40$ days (*convert to seconds*!) and $N = 40$ and $N = 960$, i.e. daily and hourly time steps, respectively. Do not, of course, show me lots of numbers. Make basic plots of the computed trajectories, i.e. the x_i, y_i values and not the v_i, w_i values. Describe in a few words what you see, and how these results relate to the local truncation error of the schemes.

c) Beyond the verification process in **P26 b)**, how can we verify these Earth-Moon calculations? That is, how does one understand how big are the errors made in these "real" computations?

One way is to measure the degree to which energy is conserved.[3] Let $U(\mathbf{r}) = -Gm_1m_2(|\mathbf{r}|)^{-1}$ be the potential energy of gravitation in this case. Then it is a theorem—*a bit of extra credit for a proof*—that the total energy, the scalar function

$$E(t) = \frac{1}{2}m_1|\mathbf{x}_1'(t)|^2 + \frac{1}{2}m_2|\mathbf{x}_2'(t)|^2 + U(|\mathbf{x}_1(t) - \mathbf{x}_2(t)|)$$

is conserved. That is, $E(t)$ is actually constant and independent of time if $\mathbf{x}_i(t)$ are the exact solutions of (1).

Start by computing the value $E(0)$ exactly from the initial conditions. (*What are the units?*) Then compute, without modifying your **P26** solvers, a 1D array `EE`, the scalar energy values $E(t)$ computed at the $N + 1$ times, from the results in **b)**. Plot the four energy curves from the four runs in one figure. Explain and comment. Describe an "energy error norm" which is small if the solution is of high accuracy, and report the values for the four runs.

d) From your computations in part **c)**, or more accurate computations for the same ODE IVP achieved by re-running the `rk4()` solver at higher resolution, how long is a lunar month?

P28. Verify that the predictor-corrector method (5.53) is second order accurate. (*Hint*: The way (5.36) is handled is an example: Combine the two formulas into one by eliminating $\hat{U}^{n+1}$. Then write down the definition of truncation error τ^n and use the fact that the exact solution satisfies $u'(t) = f(u(t))$ for any t to simplify as far as possible. Only then use Taylor's theorem, and use the fact that $u(t)$ is an exact solution when you can.)

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[2]Where I got these numbers: search "earth moon distance meters" and "mean orbital velocity moon."

[3]It would be better to address all the conserved quantities, but that is going too far for this problem.