Assignment #6

Due Monday, 20 March at the start of class

Please read Chapter 5 of LeVeque, and those parts of Appendix D which cover the matrix exponential.

P21. Check that the solution $u(t)$ given by equation (5.8) in the textbook satisfies ODE (5.6) and the initial condition $u(t_0) = \eta$. (*Hint*: To differentiate the matrix exponential you can differentiate Taylor series (D.31), in Appendix D, term by term.)

P22. The ODE IVP

$$v'' = -4v, \quad v(0) = v_0, \quad v'(0) = w_0$$

has solution $v(t) = v_0 \cos(2t) + \frac{1}{2} w_0 \sin(2t)$. (Verify this.) Determine this solution by first rewriting the ODE as a first-order system $u' = Au$. Then compute the solution $u(t) = e^{At} u(0)$ by using equation (D.30), in Appendix D.

(*Hint*: Recall A is diagonalizable if there is an invertible matrix R and a diagonal matrix Λ so that $AR = R\Lambda$ or equivalently $\Lambda = R^{-1}AR$. The diagonal entries in Λ are the eigenvalues of A and the columns of R are eigenvectors. Though R, Λ can be determined from MATLAB by the command `[R,Lambda] = eig(A)`, it is easiest in this problem to do the eigenvalue calculation by hand, after checking via MATLAB that you have the correct eigenvalues.)

P23. Let $f(u) = \log(u)$, the natural logarithm.

a) Sketch the graph of $f(u)$. Determine the best possible Lipschitz constant for this function over $2 \le u < \infty$. Is $f(u)$ Lipschitz continuous over $0 < u < \infty$?

b) Consider the ODE IVP $u' = \log(u)$, $u(0) = 2$. Explain why we know that this problem has a unique solution for all $t \ge 0$, based on the existence and uniqueness theory in subsection 5.2.1.

(*Hint*: Argue that f is Lipschitz continuous in a domain that the solution never leaves. Note that the domain $\mathcal{D}$ in section 5.2 is symmetric in u, around $\eta = 2$ in this case, but that $f(u)$ can be made nice for $u < \eta$ in manner such that the theorem can apply and the solution is unchanged.)

P24. Consider the ODE system

$$\begin{aligned} u_1' &= 2u_1, \\ u_2' &= 3u_1 - u_2 \end{aligned}$$

with some initial conditions at $t = 0$: $u_1(0) = a, u_2(0) = b$. Solve this system two ways:

a) Solve the first equation. Then insert this into the second equation to get a nonhomogeneous linear ODE for u_2. Solve this using (5.8).

b) Write the system as $u' = Au$, compute the matrix exponential, and get the solution in the form of equation (D.30). (Simplify enough to show that parts **a)**, **b)** give the same solution.)

P25. Compute the leading term in the local truncation error of these methods:

a) the trapezoidal method (5.22),

b) the 2-step BDF method (5.25).