Assignment #5

Due Wednesday 8 March, 2017 at the start of class

There will be no lectures on Monday 27 February through Friday 3 March, because I am traveling to a conference. This Assignment is designed to be done based on the slides at:

bueler.github.io/M615S17/iterative.pdf

Please read sections 4.1, 4.2 in the textbook after you read the slides.

P17. (a) Use MATLAB, etc. to compute the 2-norm condition numbers for systems LS1 and LS2 in the slides. (*Thereby confirm that these systems have unique solutions which can be well-approximated.*)

(b) Write a MATLAB function for Richardson iteration, with signature

```
function z = richardson(A,b,x0,N,omega)
```

It should return the Nth iterate $\mathbf{x}_N$ as `z`. Confirm that it works by showing you get the same $\mathbf{x}_3$ as on page 4 of the slides.

(c) How many iterations are needed to get 8 digit accuracy for LS1 with $\mathbf{x}_0 = 0$ and using the preferred value of ω? How many iterations for $\omega = 0.1$ and $\omega = 0.5$?

P18. (a) Write MATLAB functions which do N iterations of the Jacobi and Gauss-Seidel (GS) methods:

```
function z = jacobi(A,b,x0,N)
function z = gs(A,b,x0,N)
```

For each one use the entries of A directly. That is, for `jacobi()`, implement formula (5) from the slides, and for `gs()` implement (7). Your implementation of GS should use less memory than Jacobi; make sure this is clear. (*Do not split $A = D - L - U$ and store those parts; this is a waste of memory and misses the point. You may, however, check your implementation by such splitting.*)

(b) For N iterations on an $m \times m$ matrix A, how many operations (additions, subtractions, multiplications, divisions) does Jacobi require? GS? (*Your answers will be in terms of N and m.*)

(c) For each method, how many iterations are needed to get 8 digit accuracy for LS1 using $\mathbf{x}_0 = 0$? (*Clearly state what norm you are using, and how you interpret "8 digit accuracy".*) After demonstrating that GS fails on LS2, compute an explanatory spectral radius.

P19. Show that Jacobi iteration converges if A is strictly diagonally-dominant. (*Hints: Jacobi iteration converges if and only if $\rho(M) < 1$ for $M = -D^{-1}(L+U)$. So suppose $M\mathbf{v} = \lambda\mathbf{v}$ for $\mathbf{v} \neq 0$. Choose the largest-magnitude entry v_i of $\mathbf{v}$, so that $|v_i| \geq |v_j|$ for all j. Show then that $M\mathbf{v} = \lambda\mathbf{v}$, and the assumption of strict diagonal dominance, shows $|\lambda v_i| < |v_i|$ which shows $|\lambda| < 1$.*)

P20. **(a)** In the solution to **P16** (Assignment #4) I wrote a code called `poisson.m`. Using that code, or a similar starting point, write a code `fishy.m` which solves

$$u_{xx} + u_{yy} + pu_x + qu = f(x,y)$$

on the unit square, with zero boundary values, with grid spacing $\Delta x = \Delta y = h = 1/(m+1)$, all as before. Here p, q are real numbers. Decide on how you will check correctness of your code; explain your verification process in a few sentences and a figure.

(b) `fishy.m` sets up and solves a linear system $AU = F$.

i) If you fix $p = 0$, for what q values is A strictly diagonally-dominant (SDD)?
ii) If you fix $q = 0$, for what p values is A SDD?
iii) If you fix $q = -1$, for what p and h values is A SDD?

(*This part can be answered based on the FD formulas you used in part* **(a)**. *You don't need to run* `fishy.m` *to answer this part.*)

(c) Apply Gauss-Seidel (GS) to the problem solved in part **(a)**. For each of $m = 5$ and $m = 50$ find *nonzero* values p, q where Gauss-Seidel does converge and does not converge? When convergence happens, list the number of iterations to get 8 digit accuracy. (*Perhaps design your code to either use backslash or* `gs.m` *to solve $AU = F$, according to an optional argument. Part* **(b)** *will provide guidance on the rest of this part, but note that SDD is only a* sufficient *condition for convergence.*)

(d) Gauss elimination on an $k \times k$ matrix requires $\frac{2}{3}k^3$ operations. (*This is close enough to the exact count for this problem.*) On the matrices produced by `fishy.m`, at what number of iterations would GS require just as much work as Gauss elimination?