Assignment #3

Due Wednesday, 15 February at the start of class

Please read Chapter 2 of the textbook—all of it! Also read sections 3.1–3.4 to see where we are going next, namely certain PDEs.

P11. (*Inverse matrix and Green's functions.*)

a) Explicitly write out the 5×5 matrix A from (2.43), *and* the 5×5 inverse matrix $B = A^{-1}$, for the boundary value problem $u''(x) = f(x)$ with $u(0) = u(1) = 0$ for $h = 0.25$. (*You may use the computer to get B;* `format rat` *may be useful.*)

b) For $f(x) = x$, compute the solution U to the (discretized) problem in part **a)**. Sketch the solution U, by hand or by computer, *and* also sketch the five Green's functions whose sum gives this solution. (That is, show the terms in equation (2.48) on page 27.)

c) In some literature you will see the advice that the equations in a linear system should be "well-scaled," so as to get better accuracy when solving large systems. You may see the concrete advice that the equations in a finite difference scheme for $u'' = f(x)$ will be better-scaled if you multiply them by h^2. Do so to get a new matrix $\tilde{A}$. (The Dirichlet boundary conditions in the above parts are enforced by equations "$U_0 = 0$" and "$U_{m+1} = 0$;" thse should *not* be re-scaled.) In a figure, show how this change makes the columns of $\tilde{B} = \tilde{A}^{-1}$ have more uniform magnitude than those of B.

P12. (*A sometimes ill-posed boundary value problem.*)

a) Consider the following linear BVP with Dirichlet boundary conditions:

$$u''(x) + u(x) = 0 \quad \text{for } a < x < b, \qquad u(a) = \alpha, \qquad u(b) = \beta. \tag{1}$$

(*This equation arises from linearizing the pendulum equation (2.75), for example.*) Write a MATLAB/OCTAVE/etc. finite difference code to solve this problem.

b) Determine the exact solution to the problem when $a = 0, b = 1, \alpha = 2, \beta = 3$. Test your code from part **a)** using this solution, including a demonstration of convergence at the optimal rate (*which is what?*) as $h \to 0$. Generate a convergence figure.

c) Let $a = 0$ and $b = \pi$. For what values of α and β does BVP (1) have solutions? Sketch a family of solutions in a case where there are infinitely-many solutions.

d) For a sequence of refined meshes with $h = \pi \times 2^{-j}$ for $j = 2, 4, 6, 8, 10, 12$, compute the matrix A^h from part **a)** applied to the case when $a = 0, b = \pi$. (*Do not show these matrices! They are big!*) Show a figure with h versus a matrix norm $\|(A^h)^{-1}\|$. Describe what you see, and what it says about the stability of your discretization. Given what you know from part **c)**, why is A^h even invertible at all?

P13. (*Nonlinear pendulum.*) Write a program to solve the BVP for the nonlinear pendulum as discussed in the text, i.e. problem (2.77), using the Newton iteration strategy outlined in subsection 2.16.1. Reproduce Figures 2.4(b) and 2.5, for which $T = 2\pi, \alpha = \beta = 0.7$.