Basics of the stability of ODE schemes

To address stability of ODE schemes we apply them to the simplest ODE, namely

$$\frac{du}{dt} = Cu,$$

where $u(t)$ is a function of a single variable, of course. We examine when the scheme generates exponentially-decaying, or at least not exponentially-growing, solutions. By definition, the *stability region* is the region of the relevant complex plane (see below) where the scheme generates non-growing solutions.

Consider the two simplest schemes

$$\frac{U^{n+1} - U^n}{\Delta t} = CU^n, \qquad \textit{forward Euler}$$

$$\frac{U^{n+1} - U^n}{\Delta t} = CU^{n+1}. \qquad \textit{backward Euler}$$

written as simplified difference equations,

$$U^{n+1} = \Big(1 + C\Delta t\Big)U^n, \qquad \textit{forward Euler}$$

$$U^{n+1} = \left(\frac{1}{1 - C\Delta t}\right)U^n. \qquad \textit{backward Euler}$$

We analyze stability by substituting an exponential solution $U^n = (\lambda)^n$ and asking if it grows or decays. Note there is no spatial frequency k, so this analysis is simpler than von Neumann analysis of PDE schemes. Just as with the von Neumann analysis, the question is whether $|\lambda| \le 1$.

In terms of the key parameter

$$z = C\Delta t,$$

we get these equations for λ after substituting:

$$\lambda = 1 + z, \qquad \textit{forward Euler}$$

$$\lambda = \frac{1}{1 - z}. \qquad \textit{backward Euler}$$

We consider complex $z = a + ib$ because ODE schemes are often applied to second-order ODEs[1] which have complex exponential solutions, i.e. cosines and sines as solutions.

For forward Euler we have

$$|\lambda| \le 1 \qquad \iff \qquad |1 + z|^2 \le 1 \qquad \iff \qquad (a + 1)^2 + b^2 \le 1.$$

That is, in the $z = a + ib$ complex plane, the stable values of $z = C\Delta t$ are the ones that are inside a circle of radius one centered at the point $-1 + 0i$. This "stability region" is show in Figure 1.

For backward Euler, $z = C\Delta t = a + ib$ must satisfy

$$|\lambda| \le 1 \qquad \iff \qquad 1 \le |1 - z|^2 \qquad \iff \qquad 1 \le (a - 1)^2 + b^2.$$

That is, z must be outside of a circle of radius one centered at $1 + 0i$, as show in Figure 1.

Regarding the relation to PDEs, consider the heat equation. From Fourier series solutions of the heat equation we know it acts like a lot of ODEs which are essentially $du/dt = C_m u$ where C_m is different for each Four mode $m = 1, 2, \dots$ (e.g. $\sin(m\pi x)$ for our standard heat problem).

[1]These are usually written as first-order systems of ODEs.

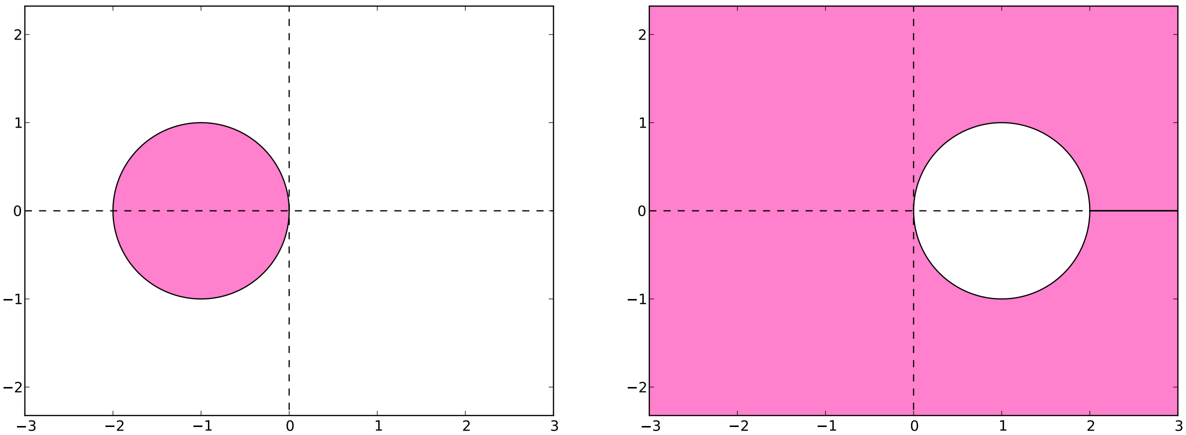

FIGURE 1. Stability regions of the complex z plane for forward Euler (left) and backward Euler (right) methods. The shaded region is where $z = C\Delta t$ gives a stable solution.

Specifically C_m is a multiple of $-m^2$ (e.g. $C_m = -\pi^2 m^2$ for our standard heat problem). Thus we care about $z = C\Delta t$ values which are on the negative real axis in the complex plane, and they are arbitrarily far along that axis.

These heat equation z-values are *all* inside the stability region for the backward Euler method, which is one way to describe or derive the unconditional stability of the implicit scheme. For the explicit scheme, by shortening the time step enough we can get all the z values which correspond to modes on the grid inside the stability region; this is conditional stability.

There are other ODE schemes than Euler and backwards Euler, of course. For example there is the Runge-Kutta order 2 scheme which, for the ODE $dy/dt = f(t, y)$ can be written $y_{n+1} = y_n + \Delta t\, f\left(t_n + \frac{1}{2}\Delta t, y_n + \frac{1}{2}\Delta t\, f(t_n, y_n)\right)$. When applied to $du/dt = cu$ this scheme simplifies to $U^{n+1} = \left(1 + z + \frac{1}{2}z^2\right) U^n$ where $z = C\Delta t$. Its stability region is shown in Figure 2, along with the regions for the Euler method and Runge-Kutta methods of orders 3 and 4; the order 2 and 4 methods are used a lot while the order 3 method is rarely used.

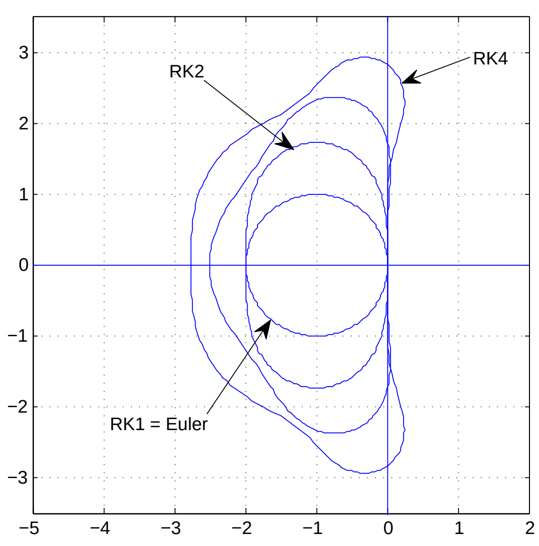

FIGURE 2. Stability regions of the complex $z = C\Delta t$ plane for Runge-Kutta methods of order 1 through 4. Regions inside the curves are stable.