Definitions and Examples Thereof

I like the style of our textbook Morton & Mayers, but it lacks formal definitions. On the first pass through this material, as you may already agree, it is a good idea to have very clear definitions. Here I have pulled them out and put them on display. I do give the page(s) in Morton & Mayers, 2nd ed (2005) on which each definition appears. This will be helpful to you in reviewing for the in-class Midterm Exam which will be on Monday, 3 March 2014.

Definition 1 (pages 16 and 17)**.** *For a given grid, the* error *(or* actual error *or* numerical error*) of a numerical scheme which computes U_j^n at a grid point (x_j, t_n) is the difference*

$$e_j^n = U_j^n - u(x_j, t_n)$$

where $u(x,t)$ is the exact solution. One can also define the maximum error at each time step*,*

$$E^n = \max_j |e_j^n|.$$

Remark. This is the error we actually care about: it measures how good the numerical result is. The truncation error below is, by contrast, something we want to be small, but we its smallness is not sufficient to make the (actual) error small. That is, instability keeps a scheme with small truncation error from giving a small (actual) error.

Definition 2 (pages 14 and 157)**.** *The* truncation error *of a numerical scheme at a point (x,t) is the amount by which the exact solution does not satisfy the scheme at that point. In particular, if the PDE satisfied by the exact solution u is written $F(u) = 0$, and if $\tilde{F}(U) = 0$ is the scheme, namely the equation satisfied by the discrete approximation U, then the truncation error is $T = \tilde{F}(u)$.*

Example. We can write the standard heat equation as

$$u_t - u_{xx} = 0$$

and the explicit scheme as

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} - \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} = 0. \tag{1}$$

Thus the truncation error is

$$T(x,t) = \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} - \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{\Delta x^2}.$$

Note the truncation error is a function of the point in the domain, and it generally varies over the domain. In proofs of convergence, however, we usually consider the worst case, the maximum over the truncation error.

Definition 3 (page 17)**.** *A* refinement path *is a sequence of positive mesh parameters* $\{(\Delta x_i, \Delta t_i)\}$, $i = 1, 2, \dots,$ *such that* $\Delta x_i \to 0$ *and* $\Delta t_i \to 0$ *as* $i \to \infty$.

Example. A refinement path for the explicit scheme (1) might be

$$\{(\Delta x_i, \Delta t_i) = ((0.2)^i, (0.01)^i)\}, \quad i = 1, 2, \dots$$

This happens to be a *stable* refinement path for the explicit scheme because $\Delta t/\Delta x^2 = (1/4)^i \leq 1/2$. But the concept of stability does not enter into the definition of "refinement path." For instance, $\{(\Delta x_i, \Delta t_i) = (0.1^i, 0.1^i)\}$ is also a refinement path. Along it, the explicit method would show instability and not converge, but the implicit method would be stable and converge.

Definition 4 (pages 15 and 157)**.** *A numerical scheme is* consistent (along a given refinement path) *if for any point* (x, t) *in the domain the truncation error goes to zero along the refinement path. A numerical scheme is* unconditionally consistent *if the truncation error goes to zero as the mesh sizes go to zero along* any *refinement path.*

Example. The truncation error of the explicit scheme (1) for the heat equation turns out to satisfy

$$|T(x,t)| \leq \frac{1}{2} M_{tt} \Delta t + \frac{1}{12} M_{xxxx} \Delta x^2,$$

where M_{tt}, M_{xxxx} are bounds on derivatives of the exact solution; see page 15. (We get this inequality from a Taylor's theorem argument, of course.) It follows that the explicit scheme is unconditionally consistent. For instance, the truncation error goes to zero along either of the paths mentioned in the previous example.

Example. Exercise 5.1 on page 190 describes an explicit three level scheme for the equation $u_t + au_x = bu_{xx}$, called the *DuFort-Frankel scheme*, which is conditionally consistent. That is, it is consistent only along *some* refinement paths.

Definition 5 (sort of defined on page 15)**.** *We say a numerical scheme has* order (of accuracy) *"*$O(\Delta t^p) + O(\Delta x^q)$*" if there exist* $C_1, C_2 > 0$ *so that*

$$|T(x,t)| \leq C_1 \Delta t^p + C_2 \Delta x^q$$

for all x, t *in the domain. (It is implied, and not usually stated, that we are considering* $\Delta t \to 0$ *and* $\Delta x \to 0$*. In some cases the constants* C_1, C_2 *might only apply for sufficiently fine grids.)*

Example. In an example above we see that the explicit scheme has order $O(\Delta t^1) + O(\Delta x^2)$. The Crank-Nicolson scheme has order $O(\Delta t^2) + O(\Delta x^2)$ (section 2.10).

Because the spatial derivative approximations are the same in these two schemes it makes sense, in the context of these two schemes, to say that the explicit scheme is "first-order accurate" and that Crank-Nicolson is "second-order accurate," but such language should be used with care. The "$O(\Delta t^p) + O(\Delta x^q)$" language communicates more because it shows both the spatial and temporal order of accuracy.

Definition 6 (pages 20–21)**.** *Suppose we have a constant-coefficient finite difference scheme $\tilde{F}(U) = 0$ for a PDE $F(u) = 0$. Suppose $U_j^n = (\lambda)^n e^{ik(j\Delta x)}$ is a wave-like solution to the scheme with (spatial) frequency k and growth/decay rate $\lambda = \lambda(k)$. The finite difference scheme is* conditionally stable *(i.e. stable along a refinement path) if there exists $K > 0$ independent of $k, \Delta t, \Delta x$ such that*

$$|\lambda(k)^n| \le K \qquad \text{for all frequencies } k \text{ and for all } n \text{ such that } n\Delta t \le t_f$$

for all $(\Delta x, \Delta t)$ on the refinement path. A finite difference scheme is unconditionally stable *if the above condition applies for every refinement path.*

A more general definition of stability is given in Chapter 5 (see page 158), but we do not use it till then.

Lemma 7 (von Neumann; page 21)**.** *If there exists $K' > 0$ independent of $k, \Delta t, \Delta x$ such that*

$$|\lambda(k)| \le 1 + K'\Delta t$$

for all k and all $(\Delta x, \Delta t)$ on the refinement path then the scheme is stable. In particular, if $|\lambda(k)| \le 1$ for all k and all $(\Delta x, \Delta t)$ on the refinement path then the scheme is stable.

Example. For the explicit scheme we calculate (page 20) that

$$\lambda(k) = 1 - 4\mu \sin^2(\tfrac{1}{2}k\Delta x),$$

where $\mu = \Delta t/(\Delta x)^2$. Thus this method is stable along any refinement path for which $\mu \le 1/2$ because $|\lambda(k)| \le 1$ for such μ. For the implicit scheme we calculate (page 26)

$$\lambda(k) = \frac{1}{1 + 4\mu \sin^2(\frac{1}{2}k\Delta x)}.$$

It follows that $|\lambda(k)| \le 1$ for any μ, so the implicit scheme is unconditionally stable.

Example. Let's apply the explicit scheme to a modified heat equation,

$$u_t = u_{xx} - u \qquad \to \qquad \frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} - U_j^n.$$

If $U_j^n = (\lambda)^n e^{ik(j\Delta x)}$ is a solution then

$$\lambda(k) = 1 - 4\mu \sin^2((1/2)k\Delta x) \, - \, \Delta t.$$

If $\mu = 1/2$ and $k\Delta x = \pi$ then $\lambda(k) = -1 - \Delta t$ so we have $|\lambda(k)| > 1$ for this worst case k. This example illustrates von Neumann's lemma above, however, because there is $K > 0$ so that $|\lambda(k)^n| \le K$ for all k and all n so that $n\Delta t \le t_f$. In fact, $K = e^{t_f}$ works because

$$|\lambda(k)^n| \le (1 + \Delta t)^n \le (1 + \Delta t)^N = (1 + t_f/N)^N \le e^{t_f}$$

where $N = t_f/\Delta t$. Thus the use of a refinement path with $\mu = 1/2$ does not generate instability even though the "sawtooth" mode (maximum gridded frequency mode) does grow in magnitude, but slowly.

Definition 8 (pages 16 and 157)**.** *A numerical scheme is* convergent (along a refinement path $(\Delta x_i, \Delta t_i)$) *if, for any fixed point* (x^*, t^*) *in the domain,*

$$\Delta x_i \to 0,\ \Delta t_i \to 0,\ x_j \to x^*,\ \textit{and}\ t_n \to t^* \qquad \textit{implies} \qquad U_j^n \to u(x^*, t^*).$$

As long as the exact solution is continuous it suffices to show that that the error goes to zero along the refinement path, that is,

$$\Delta x_i \to 0\ \textit{and}\ \Delta t_i \to 0 \qquad \textit{implies} \qquad e_j^n \to 0.$$

Example. On pages 16–17 it is shown that *if* $\mu \le 1/2$ then the global error for the explicit method for the heat equation satisfies

$$\max_{(x_j, t_n) \in \text{ grid}} |e_j^n| \le \frac{1}{2} \Delta t \left[M_{tt} + \frac{1}{6\mu} M_{xxxx} \right] t_f,$$

where M_{tt}, M_{xxxx} are bounds on derivatives of the exact solution. Assuming these bounds exist and assuming we have a refinement path for which $\mu \le 1/2$, it follows that the explicit method converges.

Example. Theorem 2.2 on page 34 shows that the θ-methods for $0 \le \theta \le 1$ converge *if* $\mu(1 - \theta) \le \frac{1}{2}$. In fact some of the other cases also converge, though we don't have a proof of that in this textbook. For example we know that if $\theta = \frac{1}{2}$ then the method is Crank-Nicolson and convergence actually occurs for any μ, but we don't have the needed tools to prove it.

Example. All the previous examples have been about the heat equation. Consider the simple advection equation

$$u_t + a(x, t) u_x = 0$$

where $a(x, t)$ gives the transport velocity. The argument in section 4.3 (pages 94–96) shows that the first order upwind scheme converges if the "CFL" condition applies; see also section 4.2. That is, convergence for this scheme occurs if

$$(\max_{x,t} |a(x, t)|) \frac{\Delta t}{\Delta x} \le 1.$$