Assignment #8

Due *Friday 18 April, 2014.*

This assignment is related to section 2.17 in the textbook Morton & Mayers, and it also relates to the ice sheet problems I work on for research. However, it is quite self-contained. No particular facts from section 2.17 are needed to do what is below.

Problem **1** is worth 30 points, but problem **2** is only worth 5 points.

1. The *porous medium equation* in one spatial dimension[1] is the nonlinear PDE

$$u_t = \left(u^2\, u_x\right)_x. \tag{1}$$

Up to constants, this equation models the spread of a gas of pressure u injected into something like packed sand. The sand has space for the gas (positive porosity) but it resists its motion (finite permeability). At the fine scale we would see turbulence of the gas in each small space, so that the gas can't spread fast at the locations where it has small pressure. The missing constants are related to the porosity of sand, the permeability of the sand, the density of the gas, and the dynamic viscosity of the gas.

Section 2.17 covers equations of the form $u_t = b(u)u_{xx}$. Equation (1), though nonlinear, is in "self-adjoint form", thus in the general class

$$u_t = \left(p(u, x, t)u_x\right)_x. \tag{2}$$

We regard (1), and (2) generally, as nonlinear versions of the linear equation $u_t = (p(x, t)u_x)_x$ covered on pages 50–51 of the textbook (section 2.15). We say $p(u, x, t) = u^2$ in (1), which happens not to depend on x or t, is the *nonlinear diffusivity*.

Note that when $u = 0$ the diffusivity $p = u^2$ is zero! Thus the porous medium equation is a *degenerate* diffusion equation because the diffusion will stop happening if $u \to 0$. There are consequences of degeneracy:

- a degenerate diffusion equation can have solutions which have finite extent; i.e. at each time t the spatial set where $u(x, t) > 0$ is finite, and
- the solution can be non-smooth, especially showing unbounded derivatives, even when the other data of the problem (e.g. boundary conditions, additional source terms, or additional coefficients) are smooth.

The second consequence implies that otherwise good numerical methods can make very large, and basically unavoidable, errors.

[1] Actually I should say it is *a* porous medium equation with exponent $m = 3$. One can write it as $u_t = (1/m)(u^m)_{xx}$. Any equation of the form $v_t = C(v^m)_{xx}$ is called a "porous medium equation" if $m > 1$. This is a nonlinear and degenerate diffusion. If $m = 1$ this equation is the linear diffusion or heat equation. If $m < 1$ this equation is called the "fast diffusion equation," a nonlinear but non-degenerate diffusion. Mathematical references for these PDEs are J. Vázquez, *The Porous Medium Equation*, Oxford Press, 2007 and L. Evans, *Partial Differential Equations*, 2nd ed., AMS Press 2010.

The above is just preamble. Here is what I would like you to do:

a) Let $u_0 > 0$ be a constant. I claim that

$$u(x,t) = \begin{cases} \frac{1}{t^{1/4}}\left(u_0^2 - \frac{x^2}{4t^{1/2}}\right)^{1/2}, & |x| < 2\,u_0\,t^{1/4}, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

is an exact solution to (1). Check this fact using by-hand differentiation. This solution is called Barenblatt's solution.[2]

b) Sketch $u(x,1)$ from part **a)** by hand. Then, in a neat code of at most 20 lines, make a movie of the solution $u(x,t)$ in part **a)**, for instance showing it on the interval $-10 \le x \le 10$ from $t = 0.1$ to $t = 10$. Use $u_0 = 2$. Generate at least 100 frames. Show me the code; don't bother printing frames from the movie.

c) I propose the following explicit scheme for (1):

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{p_{j+1/2}\left(U_{j+1}^n - U_j^n\right) - p_{j-1/2}\left(U_j^n - U_{j-1}^n\right)}{\Delta x^2}, \tag{4}$$

where

$$p_{j+1/2} = \left(\frac{U_j^n + U_{j+1}^n}{2}\right)^2.$$

What would you expect the stability criterion to be? Describe how to determine the time step in an adaptive time-stepping method.

d) Implement scheme (4) using your adaptive time-stepping method. Use the fixed interval $-10 \le x \le 10$. Use the $t = 0.1$ and $u_0 = 2$ value of the Barenblatt solution in part **a)** as the initial condition, and also use $t_0 = 0.1$ as the initial time in your simulation. Numerically approximate the solution at $t_f = 10$.

e) Of course, the particular problem approximately solved in part **d)** has an exact solution, namely the Barenblatt solution (3). For a run with $J = 20$ subintervals, show the error $|U_j^N - u(x_j, t_N)|$ at the final time $t_N = t_f$ as a graph (i.e. error on the y-axis and x on the x-axis). Do the same for a $J = 100$ run, showing it in the same figure. Where is the error big? Why? (*Write a few sentences on these questions.*)

f) Show, on a single graph with log-log axes, how the average and maximum final-time errors decrease as $\Delta x \to 0$. Specifically, do four or five runs, say with $J = 20, 40, 100, 200, 400$, or more, to generate this graph. Describe quantitatively the rates at which these errors go to zero. What would the rates be on a non-degenerate diffusion problem?

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[2] G. Barenblatt (1952). *On some unsteady motions of fluids and gases in a porous medium.*

2. Please look at these PDF slides:[3]

https://github.com/bueler/karthaus/blob/master/slides.pdf?raw=true

Note that on some slides there are "movies"—see slides 23 and 41—so perhaps use Acrobat reader if the movies are not running with another PDF viewer. Please browse through slides 1–53; the questions below address those, but you can go further if you find it amusing.

These are slides I use to teach a three-hour course in numerical methods for ice sheet simulations at a very small town in northern Italy called "Karthaus".[4] The audience consists of graduate students who already have a research focus on glaciers, ice sheets, and climate. Many in the audience have already taken a course like Math 615.

There are topics on these slides that are clearly outside the scope of Math 615! But I claim that you will see many familiar things, including a mini-course in finite differences, the heat equation, and diffusion equations. On the other hand, the slides lead to a basic simulation of the flow in the Antarctic ice sheet, and to a model of ice shelves. So directing you to these slides is an attempt to show a "real" application of the methods from Math 615.

Now, all I am *actually* asking you to do for this problem is to answer this incredibly-shallow, short-answer, five-question *QUIZ* on the slides:

i) Give the title of the slide which has a movie which shows the same kind of thing as on Morton & Mayers, pages 68-69.

ii) There is a slide which compares the "shallow ice approximation" (SIA) equation and the heat equation side-by-side. Give the formula for the SIA diffusivity, which comes from this comparison.

iii) Ice sheets have what four outstanding properties as fluids?

iv) On the slide titled "avoid the instability", the last inequality may surprise you because it has "$\frac{1}{4}$" on the right. Find this same inequality in Morton & Mayers, and explain why "$\frac{1}{4}$" is correct.

v) For the SIA there is a solution analogous to the Barenblatt solution. Who found it and when?

[3] The cool kids all have github.com sites. As such a site is free, this is a low barrier to being cool. Except you'll want to learn `git`.

[4] The 5000 year old "ice man" Ötzi was found near Karthaus; google it . . .