Assignment #6

Due *Friday March 28, 2012.*

Read sections 2.13, 2.14, 2.15 of Morton & Mayers.

1. Let $x_{j+1/2} = x_j + \Delta x/2$ and $p_{j+1/2} = p(x_{j+1/2})$. Show that the "staggered grid" explicit scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{p_{j+1/2}(U_{j+1}^n - U_j^n) - p_{j-1/2}(U_j^n - U_{j-1}^n)}{\Delta x^2}$$

for the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right)$$

is consistent if we also assume that $p(x)$ has a continuous derivative.

(*Hints*: This replaces Exercise 2.7, page 59. This question is easier. You seek the leading terms in the truncation error. Use Taylor's theorem to get

$$p(x+\epsilon)\left[u(x+\Delta,t) - u(x,t)\right] = \big(p(x) + p'(\xi)\epsilon\big)\left[u_x(x,t)\Delta + \frac{1}{2}u_{xx}(x,t)\Delta^2 + \frac{1}{6}u_{xxx}(\nu,t)\Delta^3\right].$$

Now use this twice, for $\epsilon = \pm\Delta x/2$, to expand and simplify the expression

$$p(x+\Delta x/2)\left[u(x+\Delta x,t) - u(x,t)\right] - p(x-\Delta x/2)\left[u(x,t) - u(x-\Delta x,t)\right].$$

Divide by Δx^2. Also expand the finite difference approximation of the time derivative, and then state the truncation error. Finally consider what happens as $\Delta t \to 0$ and $\Delta x \to 0$.)

2. (*This is a simplified version of Exercise 2.9 on page 60.*) Consider application of the θ-method to approximate the equation $u_t = u_{xx}$ with the choice

$$\theta = \frac{1}{2} + \frac{(\Delta x)^2}{12\Delta t} = \frac{1}{2} + \frac{1}{12\mu}.$$

Show that

i) the resulting scheme is unconditionally stable,

ii) the scheme has truncation error $O(\Delta t^2) + O(\Delta x^2)$, and finally

iii) the scheme provides rather more damping for all modes that oscillate from time step to time step than does the Crank-Nicolson scheme.

On *iii)*, make sure you clearly identify "all Fourier modes that oscillate from time step to time step".

3. Consider the linear but variable-coefficient heat equation problem

$$u_t = b(x,t)\,u_{xx} + C, \qquad u(0,t) = 0, \quad u(1,t) = 0, \qquad u(x,0) = f(x).$$

This is the problem addressed at the beginning of section 2.15. (*Which you should read and contemplate!*) The coefficient $b(x,t)$ is the *diffusivity* of this heat problem. Here I propose an "adaptive-time-stepping explicit" method for this problem.

i) Suppose that at time t_n the next time step Δt_n is determined by the criterion

$$\frac{\Delta t_n}{(\Delta x)^2}\left(\max_j b(x_j, t_n)\right) \le \frac{1}{2},$$

and this determines the next time $t_{n+1} = t_n + \Delta t_n$. Explain in a sentence or two how this differs from (2.132).

ii) Suppose

$$b(x,t) = \frac{1}{30}(2 + \sin(4\pi x)) + 3e^{-30(t-2)^2}$$

for $0 \le x \le 1$ and $0 \le t \le 3$. Give a surface plot of $b(x,t)$. I claim that around some particular time there is a sudden increase in diffusivity: identify that time.

iii) Describe in a sentence or two what should happen if an explicit method with the criterion in *i)* is applied to solve the heat problem using $b(x,t)$ from *ii)*. Specifically, what will the time steps do? Also, explain in a sentence or two why an unconditionally-stable implicit method used with large time steps might actually miss important effects.

iv) Implement the adaptive-time-stepping explicit scheme. Specifically, write a MATLAB code which solves the heat problem, using $b(x,t)$ from *ii)*, by the explicit method (2.130), and using the time stepping criterion from part *i)*. Use $u(x,0) = x(1-x)$, $C = 1/3$, and try both $J = 10$ and $J = 40$ spatial grids. Show the solution $u(x,t)$ at $t = 1, 2, 3$ for each of the two spatial grids, and show what happens to the time steps.