Assignment #4

Due *Monday, 24 February 2012*.

Read sections 2.7, 2.8, 2.10, 2.11, and 2.12 of MORTON & MAYERS. The book is dense, I know. You will be rereading sections multiple times!

Browsing section 2.9 *is* a good idea. However, throughout this course we will assume that MATLAB is a completely-adequate tool for linear algebra. We will not dig deeper than that, as it is a whole other course.[1] In any case, *don't hand-code the Thomas algorithm in section 2.9 except as an exercise for yourself*. My claim: for linear algebra and ODE initial value problems always use professionally-written codes!

1. Exercise 2.3 in MORTON & MAYERS (page 58).

2. Exercise 2.6 in MORTON & MAYERS (page 59). Do only parts (i) and (ii). (*The analysis in part* (iii) *is similar to that in* (ii)*, so let's avoid the extra work.*) The result from part (i) is used in part (ii). A good idea is to draw the stable region in the b, c plane in part (i).

THERE IS ANOTHER PROBLEM ON THE BACK!

[1] To learn about numerical linear algebra, this textbook is highly recommended: Trefethen and Bau, *Numerical Linear Algebra*, SIAM Press 1997.

3. We can consider applying the explicit and implicit methods to a heat equation with constant conduction $K > 0$ and an additional "reaction" term with constant rate C. That is, the PDE is

$$u_t = K\,u_{xx} + C\,u.$$

I claim that all cases $C > 0$, $C = 0$, $C < 0$ are reasonable to consider (i.e. useful to some application).

Here are three schemes for this PDE:[2]

EXPLICIT SCHEME:
$$\frac{\Delta_{+t}U_j^n}{\Delta t} = K\,\frac{\delta_x^2 U_j^n}{\Delta x^2} + C\,U_j^n$$

A SEMI-IMPLICIT SCHEME:
$$\frac{\Delta_{+t}U_j^n}{\Delta t} = K\,\frac{\delta_x^2 U_j^{n+1}}{\Delta x^2} + C\,U_j^n$$

FULLY-IMPLICIT SCHEME:
$$\frac{\Delta_{+t}U_j^n}{\Delta t} = K\,\frac{\delta_x^2 U_j^{n+1}}{\Delta x^2} + C\,U_j^{n+1}$$

(a) Apply the Fourier/von Neumann stability analysis of section 2.7 to each of the schemes and discuss the results. Will the semi-implicit and fully-implicit schemes be very different in terms of their stability? Explain.

(b) Implement in MATLAB the semi-implicit scheme for this problem:

$$u_t = u_{xx} + 2u, \qquad u(0,t) = 0,\ u(\pi,t) = 0, \qquad u(x,0) = \sin(x) + \sin(3x).$$

The exact solution of this problem is

$$u(x,t) = e^t \sin(x) + e^{-7t}\sin(3x).$$

Using $\Delta x = \pi/100$ and $\Delta t = 0.01$, measure the numerical error at $t_f = 0.5$. In what you turn in, of course you should include the implementation (= the code) and also give reasonable, brief evidence of success.[3]

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[2]Just an observation: In the $K = 0$ case the PDE $u_t = K\,u_{xx} + C\,u$ would become the ODE $\dot{u} = Cu$. The three schemes become only two schemes for the ODE: explicit and semi-implicit become (forward) Euler and fully-implicit becomes backward Euler.

[3]Since you know the exact solution, just plotting $u(x, t_f)$ and its numerical approximation is a good idea, but not much evidence of success. Neither is just measuring the error on one grid, though I asked you to do that. But evidence that the grid decays, and even decays at the expected rate, as $\Delta t \to 0$ and $\Delta x \to 0$, *is* evidence. So, show me one good plot that *is* evidence of success!