Assignment #3

Due *Friday, 14 February 2014.*

Read sections 2.5, 2.6, and 2.7 of Morton & Mayers, 2nd ed.

1. On page 38 there is a finite difference scheme for the heat equation $u_t = u_{xx}$, namely equation (2.98). This question is about (2.98), and you do not need to understand (2.99) and later formulas.

(a) Explain in two sentences how (2.98) is different from the explicit scheme (2.19), including about the differences in implementing the two schemes. You might compare the stencils of the two schemes, and consider how to start the numerical computation from the initial condition $u(x, 0) = u^0(x)$

(b) Implement (2.98) by writing a Matlab code that uses it to solve the problem

$$u_t = u_{xx}, \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = \sin(2\pi x).$$

This problem has exact solution $u(x,t) = e^{-4\pi^2 t}\sin(2\pi x)$, a fact you can use to evaluate (informally, for now) how well your code is doing. *Hint. Even the best intentions and the best programming practices will yield* undesirable *results, so keep reading.*

(c) Simplify the truncation error

$$T(x,t) := \frac{u(x,t+\Delta t) - u(x,t-\Delta t)}{2\Delta t} - \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2}$$

for method (2.98) on page 38. In particular, show that it satisfies

$$T(x,t) = Au_{ttt}(x,\tau)\Delta t^2 - Bu_{xxxx}(\xi,t)\Delta x^2$$

for some $x - \Delta x \le \xi \le x + \Delta x$ and $t - \Delta t \le \tau \le t + \Delta t$. Supply specific numbers for A and B. Thus, assuming that u_{ttt} and u_{xxxx} are bounded, we see that $T(x,t) = O(\Delta t^2) + O(\Delta x^2)$. Note this *better* truncation error than for (2.19). *Hint: You will use the fact "$u_t = u_{xx}$", a property of the exact solution $u(x,t)$.*

Comment: The explicit three-level scheme (2.98) is *unconditionally unstable*. DO NOT USE IT! It is both significantly more accurate than the simple explicit method, in the sense of truncation error, *and* completely useless. When we get to section 2.12 I'll remind you of method (2.98). We will show that it is unstable just by a hand calculation.

Nonethless one can hardly blame Richardson for using it in 1918 or so, for the purpose of numerical weather prediction, about 30 years before the first electronic computer.

2. Compute the truncation error of the fully-implicit scheme (2.63), with stencil shown in figure 2.5, for the heat equation $u_t = u_{xx}$. Is this scheme consistent? Unconditionally consistent? *Hint. To compute the truncation error you will use the fact that the solution $u(x,t)$ satisfies the PDE $u_t = u_{xx}$ at some point. Which point?*

3. In class, showed that formulas (2.11) and (2.13) together solved the standard heat problem (2.7), (2.8), (2.9). This problem asks you to work with the analogous vibrating string solution. (*Comment: If you have taken a course in PDEs, this is a review. In any case, this is supposed to be an easy problem.*)

(a) Show by substitution that

$$u(x,t) = \sum_{m=1}^{\infty} \left[a_m \cos(cm\pi t) + b_m \sin(cm\pi t)\right] \sin(m\pi x)$$

solves (standard) vibrating string problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad \text{for } t > 0, \quad 0 < x < 1, \\ u(0,t) &= u(1,t) = 0, \qquad \text{for } t > 0, \\ u(x,0) &= F(x), \qquad \text{for } 0 < x < 1, \\ u_t(x,0) &= G(x), \qquad \text{for } 0 < x < 1. \end{aligned}$$

Also give, with appropriate explanation, formulas for the Fourier coefficients a_m and b_m in terms of integrals of $F(x)$ and $G(x)$.

(b) Show by substitution that D'Alembert's solution[1]

$$u(x,t) = \frac{1}{2}\left[F(x+ct) + F(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(z)\, dz$$

solves the wave equation problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad \text{for } t > 0, \quad -\infty < x < \infty, \\ u(x,0) &= F(x), \qquad \text{for } -\infty < x < \infty, \\ u_t(x,0) &= G(x), \qquad \text{for } -\infty < x < \infty. \end{aligned}$$

Note there are no ends to the real line, and thus no boundary conditions.

(c) Using MATLAB, plot D'Alembert's solution if $c = 2$, $F(x) = e^{-x^2}$, and $G(x) = 0$. In particular, show $u(x,t)$ at $t = 0, 1, 3$, all on the interval $-10 < x < 10$ and in the same plot.

[1]D'Alembert (1747), *Recherches sur la courbe que forme une corde tendu mise en vibration* [Research on the curve that a tense cord forms when set into vibration]. This was the first thing ever done with a PDE, by my understanding.