Assignment #8

Due *Monday, 23 April 2012.*

Read subsections 4.1–4.9, 4.11, and 4.12 of Morton & Mayers, 2nd ed.. (We will do nothing related to 4.10.)

1. *This paragraph is explanation only*: Though the FTCS scheme (4.17) is an obvious one to try, we learned in class (von Neumann analysis) that it is unstable on any refinement path where $\nu = a\Delta t/\Delta x$ is constant. This prompted early (1950s) researchers to look for alternative schemes. Upwind is one, but it has only $O(\Delta t^1 + \Delta x^1)$ accuracy even if the solution is smooth. Another is the Lax-Wendroff scheme, which is $O(\Delta t^1 + \Delta x^2)$ when the solution is smooth. Another is the $O(\Delta t^1 + \Delta x^2)$ Lax-Friedrichs scheme here, but it is not used much in practice because Lax-Wendroff is better. Specifically Lax-Friedrichs is more damping and dispersive (i.e. more than is desirable). By the way, Peter Lax is still alive, and published lots in the last decade.

(a) Consider the Lax-Friedrichs scheme which replaces "U_j^n" in FTCS by the average of neighbors:

$$\frac{U_j^{n+1} - \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right)}{\Delta t} + a\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0.$$

Let $\nu = a\Delta t/\Delta x$. Assuming a is constant, apply the von Neumann analysis and show that the scheme is von Neumann stable if $|\nu| \le 1$, that is, if CFL applies.

(b) Show that you can write the Lax-Friedrichs scheme as

$$U_j^{n+1} = \tfrac{1}{2}(1+\nu)U_{j-1}^n + \tfrac{1}{2}(1-\nu)U_{j+1}^n.$$

How does CFL relate to interpreting this form of the scheme as an average?

(c) Show that the truncation error of the scheme satisfies

$$|T_j^n| \le M_1\Delta t + M_2(\Delta x)^2,$$

and identify M_1, M_2. Assume that the solution $u(x,t)$ is sufficiently smooth.

2. Exercise 4.3 from Morton & Mayers, 2nd ed. There is a typo in this problem: "ξ" should be "ϵ". Note that Figure 4.9 is an example of the phenomenon being addressed here.

3. See Figure 4.8 on page 104. This figure shows the success of the Lax-Wendroff scheme (4.44) in solving an advection problem when the initial data $u(x,0)$ and the advection velocity $a(x,t)$ are smooth. Specifically, the initial-and-boundary value problem that is being solved is (4.33) and (4.45), plus a boundary condition:

$$u_t + a(x,t)u_x = 0, \qquad a(x,t) = \frac{1+x^2}{1+2xt+2x^2+x^4},$$
$$u(0,t) = 0, \qquad u(x,0) = \exp\left[-10(4x-1)^2\right].$$

The exact solution is (4.35), a function defined on $x \geq 0$, $t \geq 0$, namely:

$$u(x,t) = \begin{cases} u(x^*(x,t),0), & x^*(x,t) > 0, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{where} \qquad x^*(x,t) = x - \frac{t}{1+x^2}.$$

(a) (*These questions are intended to help you understand the nature of the exact solution, so you can understand what the numerical schemes below should be producing.*) Check that the exact solution actually solves the initial-and-boundary value problem. Compute the maximum value of $a(x,t)$ in the first quadrant (i.e. $x \geq 0$, $t \geq 0$). Draw a sketch of the first quadrant showing in which region the exact solution $u(x,t)$ is identically zero, and in which region the formula $u(x,t) = u(x^*(x,t),0)$ applies.

(b) Use the upwind scheme to produce a figure comparable to Figure 4.8. Assume that the x values shown are in the interval $[0,1]$. You will *not* need to add a boundary condition to do this.

(c) Use the Lax-Friedrichs scheme—see problem **1** above—to produce another figure comparable to Figure 4.8. You will need to add a boundary condition at the right to do this, because this numerical scheme requires it, so add "$u(1,t) = 0$".

(d) Use the Lax-Wendroff scheme to reproduce Figure 4.8 itself. Use the same numerical boundary condition as in part **(c)**.