Assignment #6

Due *Monday March 26, 2012.*

Read sections 2.14, 2.15, 2.17, 3.1, and 3.2 of Morton & Mayers.

1. (*This is a simplified version of Exercise 2.9 on page 60.*) Consider application of the θ-method to approximate the equation $u_t = u_{xx}$ with the choice

$$\theta = \frac{1}{2} + \frac{(\Delta x)^2}{12\Delta t} = \frac{1}{2} + \frac{1}{12\mu}.$$

Show that

i) the resulting scheme is unconditionally stable,

ii) the scheme has truncation error $O((\Delta t)^2 + (\Delta x)^2)$, and finally

iii) the scheme provides rather more damping for all Fourier (= von Neumann) modes that oscillate from time step to time step than does the Crank-Nicolson scheme.

(On *iii)*, clearly identify "all Fourier modes that oscillate from time step to time step".)

2. Consider the linear but variable-coefficient heat equation problem

$$u_t = b(x,t)\, u_{xx} + C, \qquad u(0,t) = 0, \quad u(1,t) = 0, \qquad u(x,0) = f(x).$$

This is the problem addressed at the beginning of section 2.15. (Which you should read, and reread, several times.) The coefficient $b(x,t)$ is the *diffusivity* of this heat problem. Here I want to suggest an "adaptive-time-stepping explicit" method for this problem.

i) Suppose that at time t_n the next time step Δt_n is determined by the criterion

$$\frac{\Delta t_n}{(\Delta x)^2}\left(\max_j b(x_j, t_n)\right) \le \frac{1}{2},$$

and this determines the next time $t_{n+1} = t_n + \Delta t_n$. Explain in a sentence or two how this differs from (2.132).

ii) Suppose

$$b(x,t) = \frac{1}{30}(2 + \sin(4\pi x)) + 3e^{-30(t-2)^2}$$

for $0 \le x \le 1$ and $0 \le t \le 3$. Give a surface plot of $b(x,t)$. Justify the description "around a particular time there is a sudden increase in diffusivity" and give the particular time?

iii) Describe in a sentence or two what should happen if an explicit method with the criterion in *i)* is applied to solve the heat problem using $b(x,t)$ from *ii)*. Specifically, what will the time steps do? Also, explain in a sentence why an unconditionally-stable implicit method might miss important effects.

iv) Implement the adaptive-time-stepping explicit scheme. Specifically, write a Matlab/Octave/pylab code which solves the heat problem, using $b(x,t)$ from *ii)*, by the explicit method (2.130), and using the time stepping criterion from part *i)*. Use $u(x,0) = x(1-x)$, $C = 1/3$, and try both $J = 10$ and $J = 40$ spatial subintervals. Show the solution $u(x,t)$ at $t = 1, 2, 3$ and show what happens to the time steps.

3. Exercise 3.1 in Morton & Mayers, 2nd ed (page 83).