Assignment #5

Due *Monday March 5, 2012.*

Read sections 2.9, 2.10, 2.11, and 2.13 of Morton & Mayers, 2nd ed. Review all sections of the textbook through 2.13, and review the handout on definitions.

1. Write a single MOP = Matlab/Octave/pylab code which implements the "θ"-method in section 2.10 for the standard heat equation problem, namely equations (2.7), (2.8), and (2.9) in Morton & Mayers. The value $0 \le \theta \le 1$ should be an input to your code. Show that it produces reasonable results for $\theta = 0$ and $\theta = 1$, specifically, on a very simple heat problem to which you know (and can plot!) the exact solution. Note that the *same* lines of code should be executed regardless of the value of θ, though of course θ will appear in the code. In fact, your code should not have conditional statements; no "`if` ... `else` ... `end`" structures. You may use "`A\b`" to solve the tridiagonal system; please do not clutter up your code with an implementation of the Thomas algorithm in section 2.9.

2. a. Implement in MOP the Crank-Nicolson method to solve the PDE problem

$$u_t = (2 - x^2)u_{xx} + u + f, \qquad u(0) = 0, \quad u(1) = 0 \tag{1}$$

for solution $u(x,t)$ given a "source" function $f(x,t)$. Again use "`A\b`" to solve the tridiagonal system.

b. Find the numerical solution $U_j^N \approx u(x, t_f)$ when $f(x,t) = e^{-t}$, $u(x,0) = 1 - x$, and $t_f = 1$. Use a spatial grid with $J + 1$ equally-spaced points, and a constant stepsize $\Delta t = t_f/N$, as usual. Let $\nu = \Delta t/\Delta x$ and fix $\nu = 0.2$. Do two runs, first with $J = 20$ and then with $J = 100$. Carefully explain how much you do, and do not, know about the correctness of this result. (Hint: *Among the other questions you must ask and address is*, Do I know the exact solution?)

c. Now suppose

$$f(x,t) = \sin(3\pi x)\,\left[1 + t\left(9\pi^2(2 - x^2) - 1\right)\right].$$

Check by hand that $u(x,t) = t\sin(3\pi x)$ is an exact solution to PDE boundary value problem (1) with initial value $u(x,0) = 0$. Explain how I came up with this exact solution. (Hint: *The method by which I came up with this exact solution is traditionally called "manufacturing" it.*)

d. Using the exact solution $u(x,t)$ in **b**, and the particular $f(x,t)$ given there, evaluate the maximum numerical error, using the program you wrote in part **a**, at time $t_f = 1$. Specifically, use a refinement path with $J = 20, 40, 80, 160, 320$; again fix $\nu = 0.2$. Display the error versus Δx in a clear way. Is this what you expected to see for the Crank-Nicolson method? You should see what is called "evidence of convergence;" explain this phrase in terms of the definition of convergence.