Assignment #4

Due *Monday, 27 February 2012*.

Read sections 2.7, 2.8, 2.10, 2.11, 2.12, and 2.15 of Morton & Mayers, 2nd ed.

Browsing section 2.9 is a good idea. Throughout this course, however, we will assume that MOP = Matlab/Octave/pylab is the right tool for linear algebra. We will not dig deeper than that. To learn about numerical linear algebra, this textbook is highly recommended: Trefethen and Bau, *Numerical Linear Algebra*, SIAM Press 1997.

1. Consider applying the explicit and implicit methods to a heat equation with constant conduction $K > 0$ and an additional "reaction" term with constant rate C; the PDE is $u_t = K\,u_{xx} + C\,u$. Note that all cases $C > 0$, $C = 0$, $C < 0$ are to be considered. Here are three schemes:

$$\text{Explicit Scheme:} \qquad \frac{\Delta_{+t}U_j^n}{\Delta t} = K\,\frac{\delta_x^2 U_j^n}{\Delta x^2} + C\,U_j^n$$

$$\text{Semi-implicit Scheme:} \qquad \frac{\Delta_{+t}U_j^n}{\Delta t} = K\,\frac{\delta_x^2 U_j^{n+1}}{\Delta x^2} + C\,U_j^n$$

$$\text{Fully-implicit Scheme:} \qquad \frac{\Delta_{+t}U_j^n}{\Delta t} = K\,\frac{\delta_x^2 U_j^{n+1}}{\Delta x^2} + C\,U_j^{n+1}$$

(a) First, revert to the $K = 0$ case. The PDE become an easy ODE. The three schemes become only two schemes for the ODE. State the ODE and solve it, identify these ODE schemes, and address their stability. (*Hint*: For the stability of the ODE schemes you will probably want to look up what that might mean.)

(b) Apply the Fourier/von Neumann analysis of section 2.7 to each of the schemes and discuss the result. Will the semi-implicit and fully-implicit schemes be very different in their stability behavior? Explain.

(c) Implement in MOP the semi-implicit scheme for this problem with $K = C = 1$:

$$u_t = u_{xx} + u, \qquad u(0,t) = 0,\ u(\pi,t) = 0, \qquad u(x,0) = \sin(x) + \sin(3x).$$

The exact solution of this problem is

$$u(x,t) = e^t \sin(x) + e^{-7t}\sin(3x).$$

Using $\Delta x = \pi/100$ and $\Delta t = 0.01$, measure the numerical error at $t = 0.5$. In what you turn in, of course you should include the implementation (= the code) and also give reasonable, brief evidence of success.

2. Exercise 2.6 in MORTON & MAYERS (page 59). Do only parts (i) and (ii). (*The analysis in part* (iii) *is similar to that in* (ii)*, so let's avoid too much work.*) The result from part (i) is used in part (ii). A good idea is to draw the stable region in the b, c plane in part (i).

3. Let $x_{j+1/2} = x_j + \Delta x/2$ and $p_{j+1/2} = p(x_{j+1/2})$. Show that the "staggered grid" explicit scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{p_{j+1/2}(U_{j+1}^n - U_j^n) - p_{j-1/2}(U_j^n - U_{j-1}^n)}{\Delta x^2}$$

for the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right)$$

is consistent if we also assume that $p(x)$ has a continuous derivative.

(*Hints*: This replaces Exercise 2.7, page 59. This question is easier. You seek the leading terms in the truncation error. Use Taylor's theorem to get

$$p(x+\epsilon)\left[u(x+\Delta,t) - u(x,t)\right] = \big(p(x) + p'(\xi)\epsilon\big)\left[u_x(x,t)\Delta + \frac{1}{2}u_{xx}(x,t)\Delta^2 + \frac{1}{6}u_{xxx}(\nu,t)\Delta^3\right].$$

Now use this twice, for $\epsilon = \pm\Delta x/2$, to expand and simplify this expression:

$$p(x+\Delta x/2)\left[u(x+\Delta x,t) - u(x,t)\right] - p(x-\Delta x/2)\left[u(x,t) - u(x-\Delta x,t)\right].$$

Divide by Δx^2. Also expand the finite difference approximation of the time derivative, and then state the truncation error. Finally consider what happens as $\Delta t \to 0$ and $\Delta x \to 0$.)