Assignment #3

Due *Monday, 20 February 2012.*

Read sections 2.5, 2.6, 2.7, and 2.8 of MORTON & MAYERS, 2ND ED. This Assignment addresses those sections.

1. **(a)** On page 38 there is a finite difference scheme for the heat equation $u_t = u_{xx}$, namely equation (2.98). Explain in two sentences how it is different from the explicit scheme (2.19); are there differences in accuracy and implementation? In particular, starting the numerical solution from an initial condition $u(x, 0) = u^0(x)$ requires more thought. Show that you can, nonetheless, implement it by writing a MOP code that uses (2.98) to solve the model problem

$$u_t = u_{xx}, \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = \sin(2\pi x).$$

This model problem has exact solution $u(x,t) = e^{-4\pi^2 t}\sin(2\pi x)$, a fact you can use to evaluate (informally, for now) how well your code is doing. *Hint.* Even the best intentions for this code will yield undesirable results, so keep reading.

(b) Simplify the truncation error

$$T(x,t) := \frac{u(x,t+\Delta t) - u(x,t-\Delta t)}{2\Delta t} - \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2}$$

for method (2.98) on page 38. In particular, show that it satisfies

$$T(x,t) = Au_{ttt}(x,\tau)\Delta t^2 - Bu_{xxxx}(\xi,t)\Delta x^2$$

for some $x - \Delta x \le \xi \le x + \Delta x$ and $t - \Delta t \le \tau \le t + \Delta t$. Supply numbers for A and B. Thus, under the hypothesis that u_{tt} and u_{xxxx} are bounded, we see that $T(x,t) = O(\Delta t^2, \Delta x^2)$; is this a better or worse truncation error than for (2.19)? *Hint*: You will use the fact that "$u_t = u_{xx}$" is a property of the solution $u(x,t)$, whether you know a formula for the solution or not.

(c) Make two figures, using in each figure values of Δt and Δx which would work fine for the standard explicit method (2.19), which show the instability failure of (2.98). Use the specific model problem mentioned in **(a)**.

> *Comment*: The explicit three-level scheme (2.98) is *unconditionally unstable.* DO NOT USE IT IN REAL LIFE! It is both significantly more accurate (locally!) than the simple explicit method *and* completely useless. When we get to section 2.12 I'll remind you of method (2.98), and we see, in advance of implementation, that it is unstable. But one can hardly blame L. F. Richardson, who invented it for the purpose of numerical weather prediction 30 years before the first real computer.

2. Compute the truncation error of the fully-implicit scheme (2.63), with stencil shown in figure 2.5, for the heat equation $u_t = u_{xx}$. Have you shown that this scheme is consistent? Unconditionally consistent? (Explain in a couple of sentences.) *Hint.* To compute the truncation error you will use the fact that the solution $u(x,t)$ satisfies the PDE $u_t = u_{xx}$ at some point. Which point?

3. Exercise 2.3 in MORTON & MAYERS (page 58).