Review Topics
for In-Class Midterm Exam on *Friday 22 October, 2021*

The Midterm Quiz will cover Lectures 1, 2, 3, 4, 5, 6, 7, 8, 10, 11 in TREFETHEN & BAU. (*Lecture 9 will not be covered, but feel free to state algorithms as runnable* MATLAB*!*) The problems will be of these types: state definitions, state theorems, state algorithms (as pseudocode or MATLAB), describe or illustrate basic geometrical ideas, make basic applications of theorems, make basic calculations, prove simple theorems/corollaries.

Definitions. Know how to define:

- matrix-vector product; matrix-matrix product
- adjoint; hermitian; transpose
- inner product; outer product
- unitary
- $\| \cdot \|_p$ of a vector in $\mathbb{C}^m$ for any $1 \leq p \leq \infty$
- induced matrix norm $\| \cdot \|$
- Frobenius matrix norm $\| \cdot \|_F$
- projector; orthogonal projector
- eigenvalue; eigenvector
- singular value

Matrix Factorizations and Constructions. Know the properties of the factors in each factorization below. (*For example, $\hat{U}$ has orthonormal columns and is of same size as A in the $m \geq n$ case of the reduced SVD factorization $A = \hat{U}\hat{\Sigma}V^*$.*) Assume $A \in \mathbb{C}^{m \times n}$ unless otherwise stated. Be able to use the factorization in simple calculations. Be able to compute the factorization by hand in sufficiently small and simple cases.

- full SVD: $A = U\Sigma V^*$
- reduced SVD, $m \geq n$: $A = \hat{U}\hat{\Sigma}V^*$
- full QR, $m \geq n$: $A = QR$
- reduced QR, $m \geq n$: $A = \hat{Q}\hat{R}$
- eigenvalue, $m = n$: $A = X\Lambda X^{-1}$
- orthogonal projector: $P = \hat{Q}\hat{Q}^* = A\left(A^*A\right)^{-1}A^*$
- Householder reflector: $F = I - 2vv^*/(v^*v)$

Algorithms. Be able to state these algorithms, including the amount of work to leading order.

- matrix-vector and matrix-matrix products
- Alg. 7.1: classical Gram-Schmidt for QR
- Alg. 8.1: modified Gram-Schmidt for QR
- Alg. 10.1: Householder triangularization for QR
- Alg. 10.2: compute Q^*b after Householder triangularization
- Alg. 11.2: QR for least squares on overdetermined "$Ax = b$"
- Alg. 11.1: normal equations for least squares on overdetermined "$Ax = b$"

Facts and Formulas. Know as facts. Be able to prove unless otherwise stated.

- Cauchy-Schwarz: $|x^*y| \leq \|x\| \, \|y\|$ [*proof not required*]
- invariance of $\|\cdot\|_2$ and $\|\cdot\|_F$ matrix norms under unitaries
- $\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$
- $\|A\|_2 = \sigma_1$
- rank(A) is number of nonzero singular values (in exact arithmetic)
- for $A \in \mathbb{C}^{m \times n}$ with $m \geq n$, A has full rank if and only if A^*A is nonsingular
- the singular values of A are the square roots of the eigenvalues of A^*A
- if P is a projector then so is $I - P$
- if P is an orthogonal projector then so is $I - P$, and furthermore $I - 2P$ is unitary

Ideas. Please be comfortable with the following ideas! Some ideas correspond to theorems, but otherwise it is just a perspective or paradigm.

- L1 and L2: how to think about Ax, $A^{-1}b$, Qx, Q^*b
- L4: the image of the unit sphere under any $m \times n$ matrix is a hyperellipsoid
- L5: sums like this are optimal (in what sense?) approximations of A:

$$A_\nu = \sum_{j=1}^{\nu} \sigma_j u_j v_j^*$$

- L7: construction of orthogonal functions (e.g. orthogonal polynomials) is an application of the Gram-Schmidt process, and/or of $A = QR$, when the columns of A are infinitely long
- L8: the leading-order number of operations in some algorithm can be counted by geometric arguments
- L10: Gram-Schmidt is "triangular orthogonalization" while Householder is "orthogonal triangularization"[1]

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[1] Gauss elimination is "triangular triangularization" ... but it will not be on the Midterm Exam!