Matrix Norm Essentials

- Matrix norms have vector norm properties:
  - $\|A\| \geq 0$ and $\|A\| = 0 \implies A = 0$
  - $\|A + B\| \leq \|A\| + \|B\|$
  - $\|\alpha A\| = |\alpha| \, \|A\|$

- Induced norms, and the Frobenius norm, have additional multiplicative property
  - $\|AB\| \leq \|A\| \|B\|$

- Only four norms to know:

$$\|\cdot\|_1, \quad \|\cdot\|_2, \quad \|\cdot\|_\infty, \quad \|\cdot\|_{\mathrm{Frob}}$$

- Three have easy-to-compute formulas $(1, \infty, \mathrm{Frob})$.

- Three are induced from vector norms $(1, 2, \infty)$.

- Induced norms have $\rho(A) \leq \|A\|$.
  - But expect $\rho(A) < \|A\|$ in general, and sometimes $\rho(A) \ll \|A\|$.

- $\|\cdot\|_2$ norm best for Euclidean ideas and hermitian/normal matrices. Reasons:
  - $\|QA\|_2 = \|A\|_2$ if Q is unitary ($Q^* Q = I$).
  - $\sigma_1(A) = \|A\|_2$.
  - If $A^* = A$ then $\rho(A) = \|A\|_2$.

- Iteration $v, Av, A^2 v, \ldots$ converges if and only if

$$\rho(A) < 1.$$

  - Thus if $\|A\| < 1$ then convergence. Not conversely!