Assignment #9

Due Friday, 20 November, 2015 at the start of class

Please read Lectures 14, 15, 16, 17, and 20 in Trefethen & Bau.

P18. A *circulant matrix* is one where the diagonals are constant and "wrap around":

$$C = \begin{bmatrix} c_1 & c_m & \dots & c_3 & c_2 \\ c_2 & c_1 & c_m & & c_3 \\ \vdots & c_2 & c_1 & \ddots & \vdots \\ c_{m-1} & & \ddots & \ddots & c_m \\ c_m & c_{m-2} & \dots & c_2 & c_1 \end{bmatrix} \tag{1}$$

In formulas, the entries of $C \in \mathbb{C}^{m\times m}$ are a function of row/column index differences:

$$C_{jk} = \begin{cases} c_{j-k+1}, & j \ge k, \\ c_{m+j-k+1}, & j < k. \end{cases}$$

Here $c_1, \dots, c_m$ are the entries of a column vector, the first column of C. Specifying the first column of a circulant matrix describes it completely.

Here is an extraordinary fact about circulant matrices: Every circulant matrix has a complete set of eigenvectors *that are known in advance*, without knowing the eigenvalues. Specifically, define $f_k \in \mathbb{C}^m$ by

$$(f_k)_j = \exp\left(-i(j-1)(k-1)\frac{2\pi}{m}\right) = e^{-i2\pi(k-1)(j-1)/m}, \tag{2}$$

where, as usual, $i = \sqrt{-1}$. These vectors are *waves*, i.e. combinations of familiar sines and cosines. After some warm-up exercises you will show in part **(d)** that $Cf_k = \lambda_k f_k$.

(a) Define the *periodic convolution* $u * w \in \mathbb{C}^m$ of vectors $u, w \in \mathbb{C}^m$ by

$$(u*w)_j = \sum_{k=1}^m u_{\mu(j,k)} w_k \qquad \text{where} \qquad \mu(j,k) = \begin{cases} j-k+1, & j \ge k, \\ m+j-k+1, & j < k. \end{cases}$$

Show that $u * w = w * u$.

(b) Show that $Cu = v * u$ if C is a circulant matrix and v is the first column of C.

(c) Show that for any m, the eigenvectors $\{f_1, \dots, f_m\}$ are orthogonal.

(d) For the general circulant matrix C in (1) above, confirm the "extraordinary fact" as follows: Give a formula for the eigenvalues λ_k, in terms of the entries $c_1, \dots, c_m$, by showing by direct by-hand calculation that $Cf_k = \lambda_k f_k$.

(e) Download this MATLAB function, which builds a circulant matrix with a given first column; notice how it uses the `mod()` function:

http://bueler.github.io/M614F15/matlab/circu.m

Generate the circulant matrix C with first column consisting of 20 random numbers of your choice. Use the result of **(d)** to compute the eigenvalues λ_k, and compare these against the result of `eig()`. Also, generate f_5 from (2) and check numerically that $Cf_5 = \lambda_5 f_5$.

Exercise 15.1 in Lecture 15.

Exercise 16.1 in Lecture 16.

Exercise 17.2 in Lecture 17.

Exercise 17.3a in Lecture 17.