Constructing everywhere orthogonal vector fields by complex means

Recall that in class I did an example where I started with a given vector field $\mathbf{F}(x, y)$ on the plane and used complex-variable means, namely the Cauchy-Riemann equations, to construct a second vector field $\mathbf{G}(x, y)$ satisfying

$$\mathbf{F}(x, y) \cdot \mathbf{G}(x, y) = 0 \qquad \text{for all } (x, y) \text{ in the plane.}$$

I believe I did the example correctly.

The question arose, however, what properties of $\mathbf{F}(x, y)$ were required to make it work? The following theorem substantially answers the question:

Theorem. *Suppose*

$$\mathbf{F}(x, y) = a(x, y)\,\mathbf{i} + b(x, y)\,\mathbf{j}$$

is a vector field in the plane with differentiable components a, b.

If

$$\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x} \qquad \textit{at all points in the plane}$$

then there is a function $u(x, y)$ *such that* $\nabla u = \mathbf{F}$ *at all points in the plane.*

If, in addition*, any of the following equivalent conditions hold:*

i)

$$\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} = 0$$

at all points in the plane, or

ii)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

at all points in the plane, or

iii) there exists a solution $v(x, y)$ *to the Cauchy-Riemann equations for* u *and* v,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \qquad \textit{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

then there is a vector field $\mathbf{G}(x, y)$ *such that* $\mathbf{F} \cdot \mathbf{G} = 0$ *at all points in the plane. In fact,*

$$\mathbf{G} = \nabla v = \frac{\partial v}{\partial x}\,\mathbf{i} + \frac{\partial v}{\partial y}\,\mathbf{j} = -\frac{\partial u}{\partial y}\,\mathbf{i} + \frac{\partial u}{\partial x}\,\mathbf{j}.$$

Note that if condition *i)* holds then actually a, b solve the Cauchy-Riemann equations and $\mathbf{F}$ is just a vector field version of a complex analytic function.

This first example I give now is the one I did in class, which shows how to actually construct $\mathbf{G}$ in a case where it exists.

Example. Suppose $\mathbf{F}(x, y) = 6xy\,\mathbf{i} + (3x^2 - 3y^2)\,\mathbf{j}$. Then indeed $a_y = 6x = b_x$ and we compute u from the equations $u_x = 6xy$, $u_y = 3x^2 - 3y^2$ to get $u(x, y) = 3x^2y - y^3 + c$. Now we note that $a_x + b_y = 6y - 6y = 0$ (and also $u_{xx} + u_{yy} = 6y - 6y = 0$). Thus we can find $v(x, y)$ satisfying the Cauchy-Riemann equations: $v_y = u_x = 6xy$, $v_x = -u_y = 3y^2 - 3x^2$.

From the expression for v_y, $v(x,y) = 3xy^2 + \phi(x)$. From the expression for v_x, $\phi'(x) = -3x^2$ so $\phi(x) = -x^3 + C$. Thus $v(x,y) = 3xy^2 - x^3 + C$ so $\mathbf{G}(x,y) = \nabla v = (3y^2 - 3x^2)\,\mathbf{i} + 6xy\,\mathbf{j}$. We check that $\mathbf{F} \cdot \mathbf{G} = 0$ at all points in the plane.

The proof of the above theorem is to do this example in general. That is, to see that each step of the construction of $\mathbf{G}$ is possible with the given assumptions. The crucial step in showing equivalence of conditions *i*), *ii*), and *iii*) corresponds to noting that if u is harmonic (i.e. $u_{xx} + u_{yy} = 0$) then this is a sufficient condition for there to be a v so that u and v simultaneously solve the Cauchy-Riemann equations.

The next example shows that the first "If" in the theorem holds, the second "If" may not hold, and there can then be no $\mathbf{G}$.

Example. Suppose $\mathbf{F}(x,y) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. Then $a_y = 0 = b_x$ so u exists. In fact, $u(x,y) = x^2 + y^2 + c$. Note that $a_x + b_y = 4 \neq 0$ and also that $u_{xx} + u_{yy} = 4 \neq 0$. Furthermore, *no* $v(x,y)$ *exists so that* u, v *solve the Cauchy-Riemann equations.* In fact, if such a v existed then $v_y = u_x = 2x$ so $v(x,y) = 2xy + \phi(x)$ while $v_x = -u_y = -2y$. That is, we would require $2y + \phi'(x) = -2y$ or $\phi'(x) = -4y$, an impossibility.

Note that the $\mathbf{F}$ and $\mathbf{G}$ produced by the theorem satisfy $\mathbf{F} \cdot \mathbf{G} = 0$ by construction but that they *also* satisfy $|\mathbf{F}| = |\mathbf{G}|$. Indeed this is a point made in section 20.2 of the text because, in the theorem, u and v satisfy the Cauchy-Riemann equations and $\mathbf{F} = \nabla u$ and $\mathbf{G} = \nabla v$.

Merely finding a nontrivial orthogonal vector field turns out to be not very hard and not to require any interesting conditions. The $\mathbf{G}$ which results from application of the next theorem cannot be expected to be very useful, however.

Theorem. *Suppose* $\mathbf{F}(x,y)$ *is a planar continuous vector field which is nonzero at a point* (x_0, y_0). *There is a not-everywhere-zero continuous planar vector field* $\mathbf{G}(x,y)$ *such that* $\mathbf{F} \cdot \mathbf{G} = 0$ *everywhere in the plane.*

Proof. Let $\mathbf{F} = a\,\mathbf{i} + b\,\mathbf{j}$. We seek $c(x,y), d(x,y)$ continuous scalar functions so that

$$a(x,y)c(x,y) + b(x,y)d(x,y) = 0$$

everywhere. Recalling $\mathbf{F}(x_0, y_0)$ is not the zero vector, suppose $a(x_0,y_0) \neq 0$; the other case $b(x_0,y_0) \neq 0$ is similar. Because a is continuous we know there is $\epsilon > 0$ so that $a(x,y) \neq 0$ if $\sqrt{|x-x_0|^2 + |y-y_0|^2} < \epsilon$, i.e. near (x_0,y_0). Let $d(x,y)$ be a continuous function such that $d(x_0,y_0) = 1$ and $d(x,y) = 0$ if $\sqrt{|x-x_0|^2 + |y-y_0|^2} \geq \epsilon$. Now define

$$c(x,y) = \begin{cases} -a(x,y)^{-1} b(x,y) d(x,y), & \sqrt{|x-x_0|^2 + |y-y_0|^2} < \epsilon, \\ 0, & \sqrt{|x-x_0|^2 + |y-y_0|^2} \geq \epsilon. \end{cases}$$

This function is well-defined and continuous. Furthermore, by construction, $\mathbf{F} \cdot \mathbf{G} = 0$ everywhere in the plane. In fact, where $\sqrt{|x-x_0|^2 + |y-y_0|^2} < \epsilon$ we have

$$\begin{aligned} \mathbf{F} \cdot \mathbf{G} &= a(x,y)c(x,y) + b(x,y)d(x,y) \\ &= a(x,y)\left[-a(x,y)^{-1} b(x,y) d(x,y)\right] + b(x,y)d(x,y) = 0 \end{aligned}$$

and where $\sqrt{|x-x_0|^2 + |y-y_0|^2} \geq \epsilon$ we have

$$\mathbf{F} \cdot \mathbf{G} = a(x,y)\,0 + b(x,y)\,0 = 0.$$

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