Final Exam Solutions

[*Thanks to Russell deForest for a draft of these solutions. He is not responsible for any errors found herein!*]

Problem 1. Solve the Sturm-Liouville problem $y''(x) + \lambda y(x) = 0$ on the interval $0 \le x \le L$ with boundary conditions $y'(0) = 0$ and $hy(L) + y'(L) = 0$ where $h > 0$ is constant. Identify and describe qualitatively the equation satisfied by the eigenvalues λ. Give four digit approximations of the three smallest eigenvalues when $h = 1$ and $L = 1$. Give a formula for the normalized eigenfunctions.

Solution: The given ODE has a general solution, $y(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$. Note $y'(0) = 0$ implies $0\sqrt{\lambda}A\cos(0) - \sqrt{\lambda}B\sin(0) = \sqrt{\lambda}A$. Thus $y(x) = \cos(\sqrt{\lambda}x)$; note $\lambda = 0$ is not an eigenvalue because $y(x) = Ax + B$ cannot satisfy the boundary conditions (and be nonzero). Also, $hy(L) + y'(L) = 0$ implies $h\cos\sqrt{\lambda}L - \sqrt{\lambda}\sin\sqrt{\lambda}L$ or $h\cos\sqrt{\lambda}L = \sqrt{\lambda}\sin\sqrt{\lambda}L$. Thus the eigenvalues λ are solutions of the equation, in the easiest form to graph,

$$\frac{\sqrt{\lambda}}{h} = \cot(\sqrt{\lambda}L)$$

If we plot the graphs $y = x/h$ and $y = \cot(xL)$ then the x-coordinate of each point of intersection between the graphs is the square of an eigenvalue. The figure below shows the case $h = L = 1$. One can find where the curves intersect (to four digits) by repeatedly expanding the figure around an intersection (e.g. in Matlab), or by using Newton's method based on initial guesses found from the graph:

```
>> f = @(x) x - cot(x);  fp = @(x) 1 + (csc(x))^2;
>> x=0.9
x =
   0.90000000000000
>> x=x-f(x)/fp(x)
x =
   0.85952089441179
>> x=x-f(x)/fp(x)
x =
   0.86033322774997
>> x=x-f(x)/fp(x)
x =
   0.86033358901931
>> x=x-f(x)/fp(x)
x =
   0.86033358901938
```

Thus, $\lambda_1 = 0.74017$, $\lambda_2 = 11.735$, $\lambda_3 = 41.439$, correct to the digits shown. The eigenfunction $y_n(x) = \cos(\sqrt{\lambda_n}\,x)$ corresponds to eigenvalue $\lambda = \lambda_n$. The normalized form is $\hat{y}_n(x) = \alpha_n^{-1}y_n(x)$ where $\alpha_n^2 = \int_0^L \cos^2(\sqrt{\lambda_n}\,x)\,dx = (L/2) + \lambda_n^{-1/2}\sin(2\sqrt{\lambda_n}\,L)$.

Problem 2. The partial differential equation

$$\frac{\partial u}{\partial t} = \frac{K}{2}\frac{\partial^2 u}{\partial x^2} - c\frac{\partial u}{\partial x}$$

describes diffusion of particles in a tube of fluid, where the fluid is flowing to the right at velocity c and the diffusion constant for the particles is K. $u(x,t)$ is the concentration of the particles. Choosing $K = 1$, and assuming $c \ge 0$, solve the partial differential equation by separation of variables with boundary conditions $u(0,t) = u(L,t) = 0$. Suppose the initial concentration satisfies $u(x,0) = f(x)$. Comment on the cases where $c = 0$ and where $c \gg 1$.

Solution: Let $u(x,t) = X(x)T(t)$. The PDE gives

$$\frac{T'}{T} = -\lambda^2 = \frac{1}{2}\frac{X''}{X} - c\frac{X'}{X}$$

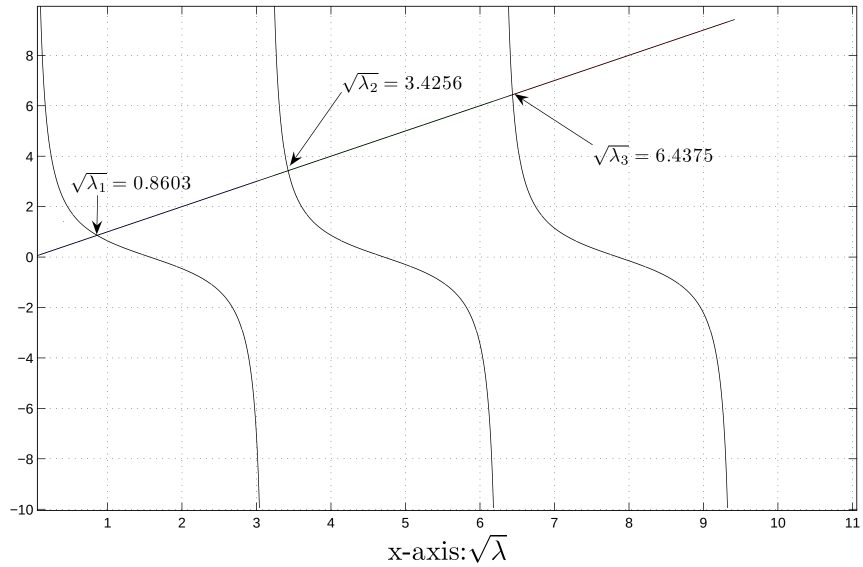

FIGURE 1. The eigenvalues λ in Problem 1 solve $x = \cot x$ where $x = \sqrt{\lambda}$.

so $(1/2)X'' - cX' + \lambda^2 X = 0$. This ODE has characteristic equation $(1/2)m^2 - cm + \lambda^2 = 0$ and solutions $m = c \pm \sqrt{c^2 - 2\lambda^2}$. If $2\lambda^2 > c^2$, a condition which is obligatory if we are to have two roots for $X(x)$ to satisfy the boundary conditions, then

$$X(x) = e^{cx}\left(A\sin(\sqrt{2\lambda^2 - c^2}\, x) + B\cos(\sqrt{2\lambda^2 - c^2}\, x)\right).$$

The boundary conditions $X(0) = 0$ and $X(L) = 0$ imply $B = 0$ and $e^{cL}A\sin(L\sqrt{2\lambda^2 - c^2}) = 0$ so $L\sqrt{2\lambda^2 - c^2} = n\pi$ or $2\lambda^2 = c^2 + (n\pi/L)^2$ or

$$\lambda_n^2 = (c^2 + n^2\pi^2/L^2)/2 \qquad \text{for } n = 1, 2, \ldots$$

Also $X_n(x) = e^{cx}\sin(n\pi x/L)$. Note that if $c = 0$ we have $\lambda_n = n\pi/(\sqrt{2}L)$ as expected, while if c is big then the lowest eigenvalue is also large.

Returning to the separated solutions, $T' + \lambda^2 T = 0$ so $T(t) = Be^{-\lambda^2 t}$. It follows that

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} e^{cx} \sin\frac{n\pi x}{L} = e^{c(x - ct/2)} \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t/(2L^2)} \sin\frac{n\pi x}{L}.$$

The coefficients a_n can be found from a Fourier sine series:

$$f(x) = u(x,0) = e^{cx}\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \qquad \text{or} \qquad f(x)e^{-cx} = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}$$

so $a_n = (2/L)\int_0^L f(x)e^{-cx}\sin(n\pi x/L)\,dx$.

If $c = 0$ then our formula for $u(x,t)$ reduces to the usual one for pure conduction in a rod with Dirichlet boundary conditions. If $|c|$ is large, however, the eigenmodes grow or decay strongly as a function of x. This means that the particles are concentrated against one of the boundaries $x = 0$ or $x = L$ as a consequence of the advection.

Problem 3(a). Show that ∇_S^2 is Hermitian on the space of continuous (and as differentiable as necessary) functions on the unit sphere with the given inner product.

Solution: Note that if $f(\theta,\phi)$ is a continuous function on the sphere then f is periodic in ϕ: $f(\theta,\phi+2\pi) = f(\theta,\phi)$. The derivatives of f are also periodic in this sense if they are continuous. These facts are essential in the integration-by-parts with respect to ϕ.

We now show that $\langle f|\nabla_S^2 g\rangle = \langle \nabla_S^2 f|g\rangle$:

$$\begin{aligned}
\langle f|\nabla_S^2 g\rangle &= \int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)^*\Big[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(g(\theta,\phi)\right)\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\left(g(\theta,\phi)\right)\Big]\sin\theta\, d\theta d\phi \\
&= \underbrace{\int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)^*\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(g(\theta,\phi)\right)\right)\right] d\theta d\phi}_{(1)} + \underbrace{\int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)^*\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\left(g(\theta,\phi)\right) d\theta d\phi}_{(2)}.
\end{aligned}$$

Integration-by-parts can be applied to (1) and (2) separately:

$$\begin{aligned}
(1) &= \int_0^{2\pi}\left(\underbrace{\underbrace{f(\theta,\phi)^*}_{=u}\underbrace{\left[\sin\theta\frac{\partial}{\partial\theta}\left(g(\theta,\phi)\right)\right]}_{=v}\Bigg|_{\theta=0}^{\pi}}_{=0,\text{ because } \sin 0=\sin\pi=0} - \int_0^{\pi}\underbrace{\sin\theta\frac{\partial}{\partial\theta}\left(g(\theta,\phi)\right)}_{v}\underbrace{\frac{\partial}{\partial\theta}\left(f(\theta,\phi)^*\right) d\theta}_{du}\right) d\phi \\
&= -\int_0^{2\pi}\int_0^{\pi}\frac{\partial}{\partial\theta}\left(f(\theta,\phi)^*\right)\frac{\partial}{\partial\theta}\left(g(\theta,\phi)\right)\sin\theta\, d\theta d\phi.
\end{aligned}$$

$$\begin{aligned}
(2) &= \int_0^{\pi}\int_0^{2\pi} f(\theta,\phi)^*\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\left(g(\theta,\phi)\right) d\phi d\theta \\
&= \int_0^{\pi}\left(\underbrace{\frac{1}{\sin\theta}f(\theta,\phi)^*\frac{\partial}{\partial\phi}\left(g(\theta,\phi)\right)\Bigg|_{\phi=0}^{2\pi}}_{=0,\text{ because } f,g \text{ are continuous on sphere}} - \int_0^{2\pi}\frac{\partial}{\partial\phi}\left(g(\theta,\phi)\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(f(\theta,\phi)^*\right) d\phi\right) d\theta \\
&= -\int_0^{\pi}\int_0^{2\pi}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(f(\theta,\phi)^*\right)\frac{\partial}{\partial\phi}\left(g(\theta,\phi)\right) d\phi d\theta.
\end{aligned}$$

Thus we have shown

$$\langle f|\nabla_S^2 g\rangle = -\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial f^*}{\partial\theta}\frac{\partial g}{\partial\theta}\sin\theta + \frac{\partial f^*}{\partial\phi}\frac{\partial g}{\partial\phi}\frac{1}{\sin\theta}\right) d\theta d\phi.$$

This expression, called the *quadratic form associated to* ∇_S^2, treats f^* and g symmetrically. Note that the second term is finite because $\lim_{\theta\to 0,\pi}\partial f/\partial\phi = 0$ because f has a well-defined value at each pole; similarly for g.

By simply exchanging the names of f, g we get $\langle g|\nabla_S^2 f\rangle^*$ in the same (quadratic) form. Thus $\langle f|\nabla_S^2 g\rangle = \langle g|\nabla_S^2 f\rangle^* = \langle \nabla_S^2 f|g\rangle$ as claimed.

Problem 3(b) [with correction]. Using the concrete formulas for Y_1^m on page 671, show that $\nabla_S^2 Y_1^m = -2\,Y_1^m$, for $m = -1, 0, +1$.

Solution: For $m=0$, $Y_1^0 = \sqrt{3/(4\pi)}\,\cos\theta$ so

$$\begin{aligned}
\nabla_S^2 Y_1^0 &= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\sqrt{\frac{3}{4\pi}}\cos\theta\right)\right) + \underbrace{\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\left(\sqrt{\frac{3}{4\pi}}\cos\theta\right)}_{=0} \\
&= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(-\sqrt{\frac{3}{4\pi}}\sin\theta\right)\right) = -2\sqrt{\frac{3}{4\pi}}\cos\theta = -2Y_1^0
\end{aligned}$$

For $m=\pm1$, $Y_1^{\pm1} = \mp\sqrt{3/(8\pi)}\,\sin\theta\exp(\pm i\phi)$ so

$$\begin{aligned}
\nabla_S^2 Y_1^{\pm1} &= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\mp\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\phi)\right)\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\left(\mp\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\phi)\right) \\
&= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left[\mp\sqrt{\frac{3}{8\pi}}\cos\theta\exp(\pm i\phi)\right]\right) + \frac{1}{\sin^2\theta}\left[\pm\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\theta)\right] \\
&= \frac{1}{\sin\theta}\left[\mp\sqrt{\frac{3}{8\pi}}\exp(\pm i\phi)\right]\left(\cos^2\theta-\sin^2\theta-1\right) \;=\; \frac{1}{\sin\theta}\left[\mp\sqrt{\frac{3}{8\pi}}\exp(\pm i\phi)\right](-2\sin^2\theta) \\
&= \mp\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\phi) = -2Y_1^{\pm1}
\end{aligned}$$

Problem 4. Using

$$P_\ell^m(0) = \frac{2^m}{\sqrt{\pi}}\frac{\cos[\frac12\pi(\ell+m)]\Gamma(\frac12\ell+\frac12 m+\frac12)}{\Gamma(\frac12\ell-\frac12 m+1)},$$

find the spherical harmonics expansion, using eqns (19.53) and (19.54), of the delta function on the equator $f(\theta,\phi)=\delta(\theta-\pi/2,\phi)$.

Solution: We need merely find the coefficients in the expansion

$$\delta(\theta-\pi/2,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}Y_\ell^m(\theta,\phi).$$

From equation (19.54),

$$a_{\ell m} = \int_{-1}^{1}\int_0^{2\pi}[Y_\ell^m(\theta,\phi)]^*\,\delta(\theta-\pi/2,\phi)\,d\phi\,d(\cos\theta) = \int_0^{\pi}\int_0^{2\pi}[Y_\ell^m(\theta,\phi)]^*\,\delta(\theta-\pi/2,\phi)\sin\theta\,d\phi\,d\theta.$$

But by definition of the delta function these integrals equal $[Y_\ell^m(\pi/2,0)]^*$. Regarding this value, equation (19.52) in the text, and the associated comment on the $m<0$ case, give

$$\begin{aligned}
[Y_\ell^m(\pi/2,0)]^* &= (-1)^m\left[\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}\right]^{\frac12} P_\ell^m(0)\cdot 1 \qquad \text{[holds for all } m \text{ because } \phi=0] \\
&= (-1)^m\left[\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}\right]^{\frac12} \frac{2^m}{\sqrt{\pi}}\frac{\cos[\frac12\pi(\ell+m)]\Gamma(\frac12\ell+\frac12 m+\frac12)}{\Gamma(\frac12\ell-\frac12 m+1)}
\end{aligned}$$

so

$$a_{\ell m} = (-1)^{m+1}\frac{2^{m-1}}{\pi}\left[\frac{(2\ell+1)(\ell-|m|)!}{(\ell+|m|)!}\right]^{\frac12}\frac{\cos[\frac12\pi(\ell+m)]\Gamma(\frac12\ell+\frac12 m+\frac12)}{\Gamma(\frac12\ell-\frac12 m+1)}.$$

Problem 5(a). For any square matrix A, define $\exp(A) = I + A + A^2/2 + A^3/3! + \cdots + A^n/n! + \cdots$. Let $\theta \in [0, 2\pi)$. Show that

$$\text{if} \quad A_\theta = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \quad \text{then} \quad \exp(A_\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Solution: Recall $\cos\theta = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}\theta^{2n}$ and $\sin\theta = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}\theta^{2n+1}$. We claim that

$$A_\theta^{2n} = (-1)^n \begin{pmatrix} \theta^{2n} & 0 \\ 0 & \theta^{2n} \end{pmatrix} \qquad \text{and} \qquad A_\theta^{2n+1} = (-1)^n \begin{pmatrix} 0 & -\theta^{2n+1} \\ \theta^{2n+1} & 0 \end{pmatrix}.$$

This suffices because it shows

$$\exp(A_\theta) = \sum_{n=0}^\infty \frac{A^{2n}}{(2n)!} + \sum_{n=0}^\infty \frac{A^{2n+1}}{(2n+1)!} = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} + \begin{pmatrix} 0 & -\sin\theta \\ \sin\theta & 0 \end{pmatrix}.$$

Use induction. Note that the claim holds for $n = 0$, and from any even power one can get the next two powers:

$$A^{2k+1} = A^{2k}A = \begin{pmatrix} (-1)^k\theta^{2k} & 0 \\ 0 & (-1)^k\theta^{2k} \end{pmatrix} \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \begin{pmatrix} 0 & (-1)^{k+1}\theta^{2k+1} \\ (-1)^k\theta^{2k+1} & \end{pmatrix},$$

$$A^{2k+2} = A^{2k+1}A = \begin{pmatrix} 0 & (-1)^{k+1}\theta^{2k+1} \\ (-1)^k\theta^{2k+1} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \begin{pmatrix} (-1)^{k+1}\theta^{2k+2} & 0 \\ 0 & (-1)^{k+1}\theta^{2k+2} \end{pmatrix}$$

Therefore the result holds for all n.

Problem 5(b). Convince yourself that $[\exp(A)]^T = \exp(A^T)$.

Solution: Since matrix addition is done entry by entry, $A = B + C$ implies $A^T = B^T + C^T$. One can check directly that $(A^2)^T = (A^T)^2$. This can be extended using induction to show that $(A^n)^T = (A^T)^n$. It follows that $[\exp(A)]^T = \exp(A^T)$.

Problem 5(c). Suppose A is a square matrix such that $A^T = -A$. Show by multiplying out the infinite series that $\exp(A)^T \exp(A) = \exp(A^T)\exp(A) = I$.

Solution:

$$\exp(A^T)\exp(A) = \exp(-A)\exp(A) = \left(\sum_{n=0}^\infty \frac{(-1)^n A^n}{n!}\right)\left(\sum_{m=0}^\infty \frac{A^n}{n!}\right) = \sum_{n=0}^\infty c_n A^n$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1}b_1 + a_n b_0$ and $a_k = (-1)^k/k!$ and $b_k = 1/k!$. Using the binomial identity, $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$ we have

$$0 = \frac{1}{n!}(1-1)^n = \frac{1}{n!}\sum_{k=0}^n \binom{n}{k}(-1)^k = \sum_{k=0}^n \frac{(-1)^k}{k!(n-k)!} = \sum_{k=0}^n \frac{(-1)^k}{k!}\,\frac{1}{(n-k)!} = \sum_{k=0}^n a_k\, b_{n-k}.$$

Thus $c_n = 0$ for all $n \geq= 1$. Also $c_0 = 1$. Thus, $\exp(A^T)\exp(A) = I$.

Problem 6(a). Show the following set of 3×3 real matrices forms a group under matrix multiplication:

$$G = \left\{ M(\theta, a, b) = \begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

Solution: (i) CLOSURE: By easy trigonometric identities,

$$\begin{aligned} M(\alpha, a_1, b_1) \cdot M(\beta, a_2, b_2) &= \begin{pmatrix} \cos\alpha & -\sin\alpha & a_1 \\ \sin\alpha & \cos\alpha & b_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta & a_2 \\ \sin\beta & \cos\beta & b_2 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) & a_2\cos\alpha - b_2\sin\alpha + a_1 \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) & a_2\sin\alpha + b_2\cos\alpha + b_1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= M(\alpha+\beta, a_2\cos\alpha - b_2\sin\alpha + a_1, a_2\sin\alpha + b_2\cos\alpha + b_1), \end{aligned}$$

which is also an element of G, so G is closed.
(ii) IDENTITY: $M(0,0,0) = I$, so $I \in G$.
(iii) ASSOCIATIVITY: Note matrix multiplication is associative; see, for example, RILEY, HOBSON, & BENCE equation (8.34) on page 258.
(iv) INVERSES: The calculation for closure above is very useful. In particular, given $M(\theta, a, b)$ we want the formula for the inverse. That is, we want to find β, c, d so that

$$M(\theta, a, b)\, M(\beta, c, d) = I = M(0,0,0).$$

So we solve

$$\begin{aligned} \theta + \beta &= 0 \\ c\cos\theta - d\sin\theta + a &= 0 \\ c\sin\theta + d\cos\theta + b &= 0 \end{aligned}$$

for β, c, d. The result is $\beta = -\theta$, $c = -a\cos\theta - b\sin\theta$, $d = a\sin\theta - b\cos\theta$:

$$M(\theta, a, b)^{-1} = M(-\theta, -a\cos\theta - b\sin\theta, a\sin\theta - b\cos\theta).$$

Therefore G is a group under matrix multiplication.

Problem 6(b). Interpret the point (x, y) in the plane as the column vector $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$. What is the effect of $M(\theta, a, b)$ on such a point? Interpret $M(\theta, a, b)$ in terms of rigid motions of the plane.

Solution: $M(\theta, a, b)$ rotates the point (x, y) around the origin by θ and then translates it by a in the x direction and by b in the y direction. In terms of rigid motions of the plane, this is a rotation of the plane by θ followed by the translation $(x, y) \mapsto (x + a, y + b)$. In particular, the inverse of this operation is naturally a translation $(x, y) \mapsto (x - a, y - b)$ followed by a rotation by $-\theta$. But as it stands that pair of operations is not in G because it has the rotation and the translation in the wrong order. This explains the nontrivial formula in "INVERSES" above.