Math 612 Mathematical Physics II (Bueler) Due *Thursday, 11 May 2006 at 5pm in my box.*

Final Exam (Take-home)

Rules. You *may* use written references of any type as long as you cite them clearly. You *may* use MATLAB and other technology to solve your problems. You *may not* talk to any person, by any means, other than me.

1. Solve the Sturm-Liouville problem

$$y''(x) + \lambda y(x) = 0$$

on the interval $0 \le x \le L$ with boundary conditions

$$y'(0) = 0 \qquad \text{and} \qquad hy(L) + y'(L) = 0$$

where $h > 0$ is constant. In particular, identify and describe qualitatively (e.g. with a graph) the equation satisfied by the eigenvalues λ. Give four digit approximations of the three smallest eigenvalues when $h = 1$ and $L = 1$. Finally, give a formula for the *normalized* eigenfunctions.

2. The partial differential equation

$$\frac{\partial u}{\partial t} = \frac{K}{2}\frac{\partial^2 u}{\partial x^2} - c\frac{\partial u}{\partial x}$$

describes (for instance) diffusion of particles in a tube of fluid, where the fluid is flowing to the right at velocity c and the diffusion constant for the particles is K. In this case $u(x, t)$ is the concentration of the particles.

Choosing $K = 1$ and assuming $c \ge 0$ for simplicity, solve the partial differential equation by separation of variables if the boundary conditions are $u(0, t) = u(L, t) = 0$ and the initial concentration satisfies $u(x, 0) = f(x)$. Comment on the cases where $c = 0$ and where $c \gg 1$.

3. (a) Recall that I defined the *Laplacian on the unit sphere* by the formula

$$\nabla_S^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

in spherical (polar) coordinates. Let us define an inner product for functions on the unit sphere by

$$\langle f|g\rangle = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} f(\theta,\phi)^* g(\theta,\phi)\, \sin\theta d\theta d\phi.$$

Show that ∇_S^2 is Hermitian on the space of continuous (and as differentiable as necessary) functions on the unit sphere with this inner product.

(b) [*ONLY THIS PART IS CORRECTED!*]Using the concrete formulas for Y_1^m on page 671, show that

$$\nabla_S^2 Y_1^m = -2\, Y_1^m, \qquad m = -1, 0, +1.$$

[*This is an example of my general claim* $\nabla_S^2 Y_l^m = -l(l+1)Y_l^m$. *It illustrates the fact that* $\lambda = -2 = -1(1+1)$ *is a degenerate eigenvalue of* ∇_S^2.]

4. I found the following formula for the associated Legendre functions $P_l^m(z)$ in Abramowitz & Stegun eds., *Handbook of Mathematical Functions*, National Bureau of Standards 1964. It is formula 8.6.1:

$$P_l^m(0) = \frac{2^m}{\sqrt{\pi}} \frac{\cos\left[\frac{1}{2}\pi(l+m)\right]\Gamma\left(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}l-\frac{1}{2}m+1\right)}.$$

Using this fact at the crucial stage, find the spherical harmonics expansion, using equations (19.53) and (19.54) in the text, of the delta function on the equator

$$f(\theta,\phi) = \delta(\theta - \pi/2, \phi).$$

[*Note the defining property of this delta function: For any continuous function* $g(\theta,\phi)$ *on the unit sphere,*

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} g(\theta,\phi)\,\delta(\theta-\pi/2,\phi)\,\sin\theta d\theta d\phi = g(\pi/2,0). \quad]$$

5. **(a)** For any square matrix A, define

$$\exp(A) = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \ldots$$

Let $\theta \in [0,2\pi)$. Show that

$$\text{if} \qquad A_\theta = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \qquad \text{then} \qquad \exp(A_\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(b) Convince yourself that $[\exp(A)]^\top = \exp(A^\top)$. [*I won't grade this part, but write out enough to see it's true.*]

(c) Suppose A is a square matrix such that $A^\top = -A$. Show by multiplying out the infinite series that

$$\exp(A)^\top \exp(A) = \exp(A^\top)\exp(A) = I.$$

[*It turns out that* A_θ *in part* ***(a)****, which has the property* $A^\top = -A$*, is not an element of a group but rather of a "Lie algebra." The exponential map takes the Lie algebra* $\mathfrak{so}(n)$*, which is the set of all* $n \times n$ *matrices such that* $A^\top = -A$*, and maps it onto the (Lie) group* $SO(n)$*. A slogan for the same idea in three dimensions is that exponentiating a cross product gives you a rotation. Regarding* ***(c)****, in fact, because it is also true that* $\det\exp(A) = e^{\operatorname{tr} A}$ *and that* $A^\top = -A$ *implies* $\operatorname{tr} A = 0$*, we see that* $\exp(A) \in SO(n)$ *if* $A \in \mathfrak{so}(n)$.]

6. **(a)** Show the following set of 3×3 real matrices forms a group under the operation of matrix multiplication:

$$G = \left\{ M(\theta,a,b) = \begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

(b) Interpret the point (x,y) in the plane as the column vector $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$. What is the effect of $M(\theta,a,b)$ on such a point? Interpret $M(\theta,a,b)$ in terms of rigid motions of the plane.