Selected Solutions to Assignment # 7

17.15. Let $L := d^2/dx^2$ be the self-adjoint linear differential operator whose domain is the set of functions with zero value at $x = 0$ and $x = \pi$. Eigenfunctions of this operator satisfy $Ly = y'' = \lambda y$, which can be made to look like Hooke's Law if written as $y'' = -(i\sqrt{\lambda})^2 y$. Solutions take the form $y(x) = A\sin(i\sqrt{\lambda}x) + B\cos(i\sqrt{\lambda}x)$, and the boundary condition $y(0) = 0$ forces $B = 0$. To satisfy $y(\pi) = 0$, the argument of the sine function must be an integer multiple of π, so $i\sqrt{\lambda} = n$ where n is any nonzero integer. It follows that the eigenvalues of L are $\lambda_n = -n^2$ with eigenfunctions $y_n(x) = A_n \sin nx$ for $n = 1, 2, \dots$. Note that $n = 0$ corresponds to the trivial solution of the eigenvalue problem. Since the operator L is Hermitian, the eigenfunctions $\{y_n\}$ are orthogonal. It is easy to show that choosing $A_n = \sqrt{2/\pi}$ makes $\{y_n\}$ orthonormal. The Green's function in terms of this basis is

$$G(x,z) = \sum_{n=1}^{\infty} \frac{y_n(x)y_n(z)^*}{\lambda_n} = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)\sin(nz)}{n^2} \tag{1}$$

A completely different expression for this same Green's function can be found by solving $L_x G = \delta(x - z)$ with boundary conditions $G(0, z) = G(\pi, z) = 0$, for all z, using the methods of section 15.2.5. (Here "$L_x G$" indicates that L is the second derivative *with respect to* x and not z.) Note that except at $x = z$, $G(x, z)$ solves $d^2y/dx^2 = 0$. Thus G has the form

$$G(x,z) = \begin{cases} A(z)x + B(z), & x \in [0, z] \\ C(z)x + D(z), & x \in [z, \pi] \end{cases} \tag{2}$$

The boundary condition imply that $A(z)\cdot 0 + B(z) = C(z)\pi + D(z) = 0$, so $B = 0$ and $D = -C\pi$. Continuity at $x = z$ imposes the condition $C(z)z + D(z) = A(z)z$, so $A(z) = C(z)(z - \pi)/z$. Finally, the jump discontinuity restriction at $x = z$ requires that $\frac{\partial G}{\partial x}\Big|_{(z-,z)}^{(z+,z)} = 1$, giving $C(z) - A(z) = 1$. Solving $A(z) = C(z)(z - \pi)/z = C(z) - 1$ yields $C(z) = z/\pi$. Substitution gives $A(z) = (z - \pi)/\pi$ and $D(z) = -\pi C(z) = -z$, and all coefficients of (2) are identified:

$$G(x,z) = \begin{cases} (z-\pi)x/\pi, & x \in [0, z] \\ zx/\pi - z, & x \in [z, \pi] \end{cases} = \begin{cases} x(z-\pi)/\pi, & x \in [0, z] \\ z(x-\pi)/\pi, & x \in [z, \pi]. \end{cases} \tag{3}$$

The functions in (1) and (3) can be shown to be the same by writing (3) in the basis of eigenfunctions of L. This is simply the Fourier Sine series representation $G(x, z) = \sum_{n=1}^{\infty} b_n(z)y_n(x)$ where $b_n(z) = \langle G(x, z)|y_n(x)\rangle$. Computing this coefficient,

$$\begin{aligned}
\sqrt{\frac{\pi}{2}}b_n(z) &= \int_0^{\pi} G(x,z)\sin(nx)\,dx = \int_0^z \frac{x(z-\pi)}{\pi}\sin(nx)\,dx + \int_z^{\pi}\frac{z(x-\pi)}{\pi}\sin(nx)\,dx \\
&= \left(\frac{z}{\pi} - 1\right)\left[\frac{-x\cos(nx)}{n}\Big|_0^z + \int_0^z \frac{\cos(nx)}{n}\,dx\right] + \frac{z}{\pi}\left[\frac{-x\cos(nx)}{n}\Big|_z^{\pi} - \int_z^{\pi} \pi\sin(nx) - \frac{\cos(nx)}{n}\,dx\right] \\
&= \frac{-z(z-\pi)\cos(nz)}{n\pi} + \frac{(z-\pi)\sin(nz)}{n^2\pi} + \frac{z(z-\pi)\cos(nz)}{n\pi} - \frac{z\sin(nz)}{n^2\pi} = \frac{-\sin(nz)}{n^2}
\end{aligned}$$

and so $b_n(z) = -y_n(z)/n^2$.

18.8. Suppose $u(x,t)$ is a function which satisfies $2u_x + 3u_t = 10$ and has constant value 3 on the line $x = t/4$. The value of $u(2,4)$ can be found by first finding a suitable starting point (x_1, t_1) at which the value of u is known, namely a point on the line $x = t/4$, and integrating along the characteristic line for the PDE. Of course, the PDE can be written as $u_t + cu_x = \frac{10}{3}$ where $c = \frac{2}{3}$. Let $x(t) = ct - x_0$ be a parameterized line in the x,t-plane. We find that $x_0 = \frac{-2}{3}$ for the line which passes through $(2,4)$. The lines $x = \frac{-2}{3} + \frac{2}{3}t$ and $x = \frac{t}{4}$ intersect at $(x_1, t_1) = \left(\frac{6}{15}, \frac{24}{15}\right)$. Note that $u(x_1, t_1) = 3$.

Define $U(t) = u(x(t), t)$. Note that $U'(t) = u_x \frac{dx(t)}{dt} + u_t$. Along the line specified, $U'(t) = 10/3$ because $dx/dt = c = 2/3$ and $u_t + (2/3)u_x = 10/3$. Integration gives $U(t) = U(t_1) + \frac{10}{3}(t - t_1)$. Then $U(t) = u((x(t), t) = u(x_1, t_1) + \frac{10}{3}\left(t - \frac{24}{15}\right) = 3 + \frac{10}{3}\left(t - \frac{24}{15}\right)$ so $u(2,4) = 3 + \frac{10}{3}\left(4 - \frac{24}{15}\right) = 11$.

19.2. Let $\Omega = [-a, a]^3$ be a cube with conductivity κ and no internal heat sources. If $u(x,y,z,t)$ is the temperature then u satisfies the diffusion equation $u_t = \kappa \nabla^2 u$. The claim is that

$$u(x,y,z,t) = A \cos \frac{\pi x}{a} \sin \frac{\pi z}{a} \exp\left[-\frac{2\kappa\pi^2 t}{a^2}\right].$$

is a solution. Note $u_y = u_{yy} = 0$. Furthermore

$$\begin{aligned}
u_x &= -\frac{A\pi}{a} \sin \frac{\pi x}{a} \sin \frac{\pi z}{a} \exp\left[-\frac{2\kappa\pi^2 t}{a^2}\right], \quad & u_{xx} &= -\frac{A\pi^2}{a^2} \cos \frac{\pi x}{a} \sin \frac{\pi z}{a} \exp\left[-\frac{2\kappa\pi^2 t}{a^2}\right] = -\frac{A\pi^2}{a^2} u \\
u_z &= \frac{A\pi}{a} \cos \frac{\pi x}{a} \cos \frac{\pi z}{a} \exp\left[-\frac{2\kappa\pi^2 t}{a^2}\right], \quad & u_{zz} &= -\frac{A\pi^2}{a^2} \cos \frac{\pi x}{a} \sin \frac{\pi z}{a} \exp\left[-\frac{2\kappa\pi^2 t}{a^2}\right] = -\frac{A\pi^2}{a^2} u
\end{aligned}$$

$$u_t = -\frac{2A\kappa\pi^2}{a^2} u$$

From the right column,

$$\kappa \nabla^2 u = \kappa(u_{xx} + u_{yy} + u_{zz}) = \kappa\left(-2\frac{A\pi^2}{a^2} u\right) = u_t$$

so the function u is a solution to the diffusion equation.

The left column of the above table of derivatives gives the components of the temperature gradient ∇u. Heat flow is present wherever a partial derivative fails to vanish. Heat flow *across* the x_k boundary occurs when the kth coordinate of ∇u is not zero. As u is constant with respect to y, there is no flow across $y = \pm a$. There is also no flow across $x = \pm a$ since

$$\frac{\partial u}{\partial x}(\pm a, y, z, t) = -\frac{A\pi}{a} \underbrace{\sin \frac{\pm \pi a}{a}}_{\sin \pi = 0} \sin \frac{\pi z}{a} \exp\left[-\frac{2\kappa\pi^2 t}{a^2}\right] = 0.$$

There is flow across $z = \pm a$, however. Finally, at $(\mathbf{x}_0, t_0) = \left(\frac{3a}{4}, \frac{a}{4}, a, \frac{a^2}{\kappa\pi^2}\right)$, one has $\nabla u|_{(\mathbf{x}_0, t)} = \frac{\pi A}{a}\left(0, 0, 0, \frac{e^{-2}}{\sqrt{2}}\right)$ which is flow in the z-direction with rate $\pi A e^{-2}/\sqrt{2}a$.