Selected Solutions to Assignment # 6

17.3. Let y_m, y_n be real eigenfunctions associated with $\mathcal{L}y = -(py')' - qy$ for $x \in [a, b]$, so that $\mathcal{L}y_k = \lambda_k \rho y_k$. Then

$$\int_{x_1}^{x_2} y_m \mathcal{L} y_n \, dx = -\int_{x_1}^{x_2} y_m[(py_n')' + qy_n] \, dx = \lambda_n \int_{x_1}^{x_2} y_m \rho y_n \, dx$$

and similarly for $\int y_n \mathcal{L} y_m$. Subtracting these and integrating by parts yields

$$\begin{aligned}
(\lambda_n - \lambda_m) \int_{x_1}^{x_2} \rho y_m y_n \, dx &= \int_{x_1}^{x_2} \left(y_m[-(py_n')' - qy_n] + y_n[(py_m')' + qy_m] \right) \, dx \\
&= \int_{x_1}^{x_2} \left(y_n(py_m')' - y_m(py_n')' \right) dx + \int_{x_1}^{x_2} q \underbrace{(y_n y_m - y_n y_m)}_{0} \, dx \\
&= \left[y_n(py_m') - y_m(py_n') \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} p \underbrace{(y_m' y_n' - y_n' y_m')}_{0} \, dx.
\end{aligned}$$

Let x_1 and x_2 be two successive zeros of y_m and suppose, without loss of generality, that $y_m > 0$ on (x_1, x_2). Then $y_m'(x_1) > 0$ and $y_m'(x_2) < 0$. Assume that y_n does not change signs on $[x_1, x_2]$. Since $\lambda_n - \lambda_m > 0$ and p is positive, the left side of the equation is positive. Since y_m and p are positive and y_m' has different signs at the endpoints, $[y_n(py_m') - y_m(py_n')]_{x_1}^{x_2} = y_n(py_m')|_{x_1}^{x_2}$ must be negative. This is contradicts that y_n does not change signs.

17.4 (a). We need to assume that $y(x)$ is continuous. In that case, integrating the differential equation from $x = -\epsilon$ to $x = \epsilon$ gives us a condition which yields the eigenvalues:

$$0 = \int_{-\epsilon}^{\epsilon} y''(x) + a\delta(x)y(x) + \lambda y(x) \, dx = y'(\epsilon) - y'(-\epsilon) + ay(0) + \lambda \int_{-\epsilon}^{\epsilon} y(x) \, dx,$$

so, because y continuous implies that $\lim_{\epsilon \to 0^+} \int_{-\epsilon}^{\epsilon} y(x) \, dx = 0$, we have

$$0 = y'(0^+) - y'(0^-) + ay(0), \tag{1}$$

a jump condition for the first derivative. We also have the conditions

$$y(-\pi) = 0, \qquad y(\pi) = 0, \qquad \text{and} \qquad y(0^+) = y(0^-). \tag{2}$$

The last of these is the continuity for y.

On the other hand, in the disconnected intervals $(-\pi, 0)$ and $(0, \pi)$ we know $y'' + \lambda y = 0$ so

$$y(x) = \begin{cases} A\cos(\sqrt{\lambda}(x+\pi)) + B\sin(\sqrt{\lambda}(x+\pi)), & -\pi \le x \le 0, \\ C\cos(\sqrt{\lambda}(x-\pi)) + D\sin(\sqrt{\lambda}(x-\pi)), & 0 \le x \le \pi. \end{cases}$$

Note that these functions are shifted to be easily evaluated at $\pm\pi$, respectively. Indeed, the conditions (2) imply $A = 0$, $C = 0$, and $B = -D$, respectively. We are seeking an eigenfunction, the magnitude of which is irrelevant, so we can choose $y(x) = \sin(\sqrt{\lambda}(x+\pi))$ for $-\pi \le x \le 0$ and $y(x) = -\sin(\sqrt{\lambda}(x-\pi))$ for $0 \le x \le \pi$. Jump condition (1) gives

$$0 = \cos(\sqrt{\lambda}(0-\pi))\sqrt{\lambda} + \cos(\sqrt{\lambda}(0+\pi))\sqrt{\lambda} + a\sin(\sqrt{\lambda}\pi)$$

and this simplifies to $\tan(\pi\sqrt{\lambda}) = 2\sqrt{\lambda}/a$ as claimed.

The above analysis actually requires $\lambda \ge 0$ because we solve $y''+\lambda y = 0$ by sines and cosines. Let $\lambda = +\mu^2$. The condition on λ is now $\tan(\pi\mu) = 2\mu/a$. The solutions of this equation are shown in figure 1 in the case $a = 2$.

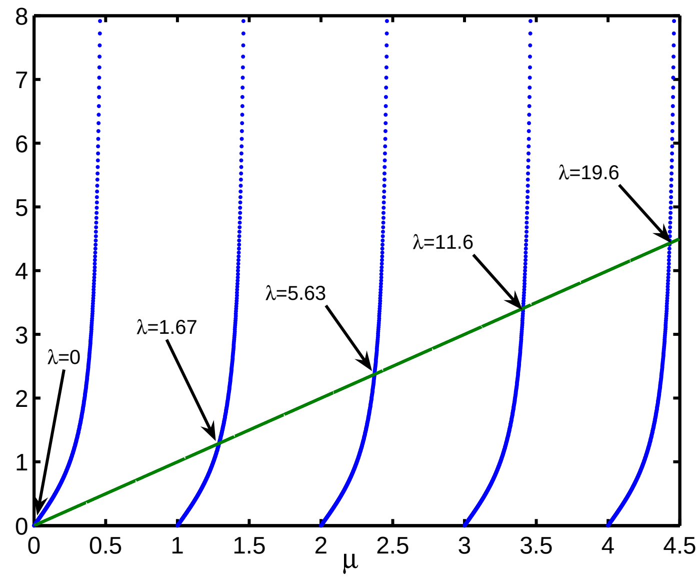

FIGURE 1. The eigenvalues solve $\tan(\pi\mu) = 2\mu/a$; case $a = 2$ shown. Note that the horizontal axis is μ but the values of $\lambda = +\mu^2$, the eigenvalues, are indicated.

(b). If $\lambda < 0$ then the solutions of $y'' + a\delta(x)y + \lambda y = 0$ are

$$y(x) = \begin{cases} A\cosh(\mu(x+\pi)) + B\sinh(\mu(x+\pi)), & -\pi \le x \le 0, \\ C\cosh(\mu(x-\pi)) + D\sinh(\mu(x-\pi)), & 0 \le x \le \pi \end{cases}$$

where $\lambda = -\mu^2$. Again $A = C = 0$ and $B = -D$ so $y(x) = \sinh(\sqrt{\lambda}(x+\pi))$ for $-\pi \le x \le 0$ and $y(x) = -\sinh(\sqrt{\lambda}(x-\pi))$ for $0 \le x \le \pi$. The jump condition indeed gives $\tanh(\pi\mu) = 2\mu/a$. Plotting a few cases indicates that there is *at most one solution*, with $\mu > 0$ and $\lambda < 0$. It only occurs if the slope of $f(\mu) = 2\mu/a$ at $\mu = 0$ is less than the slope of $g(\mu) = \tanh(\pi\mu)$ at $\mu = 0$. But this is indeed the condition $a > 2/\pi$ claimed in the hint.

17.8. Fix x and let $f(h) = e^{2hx-h^2}$. Expand in series in about $h = 0$:

$$f(0) = e^{2hx-h^2}\Big|_{h=0} = e^0 = 1, \qquad f'(0) = 2(x-h)f(h)|_{h=0} = 2x$$
$$f''(0) = 2(x-h)f'(h) - 2f(h)\big|_{h=0} = 4x^2 - 2$$
$$f'''(0) = 2(x-h)f''(h) - 2xf'(h) - 2f'(h)\big|_{h=0} = 8x^3 - 12x$$
$$f^{(iv)}(0) = 2(x-h)f'''(h) - (4x+2)f''(h) - 2f'(h)\big|_{h=0} = 16x^4 - 48x^2 + 12$$

These are the coefficients in $\sum_0^\infty \frac{H_m(x)}{n!}h^m$, and they are the first five Hermite polynomials.

Note that e^{-x^2} is an even function. For $p = 2$, $q = 3$, note $H_2(x)H_3(x)$ is an odd polynomial, so $\int_{-\infty}^{\infty} e^{-x^2}H_2(x)H_3(x)dx = 0$. For $p = 2, q = 4$, the integral is

$$\int_{-\infty}^{\infty} e^{-x^2}(64x^6 - 244x^4 - 144x^2 - 24)dx = 64\frac{6!\sqrt{\pi}}{2^6 3!} - 224\frac{4!\sqrt{\pi}}{2^4 2!} - 144\frac{2!\sqrt{\pi}}{2^2 1!} - 24\frac{\sqrt{\pi}}{2} = 0.$$

For $H_3^2(x) = 64x^3 - 192x^4 + 144x^2$, the value is $64\frac{6!\sqrt{\pi}}{2^6 3!} - 192\frac{4!\sqrt{\pi}}{2^4 2!} + 144\frac{2\sqrt{\pi}}{2^2} = 48\sqrt{\pi}$. These all match (17.52).

17.10. Let ϕ_j be orthogonalized functions, with hats representing normalized functions. Orthonormalizing monomials over the interval $[0, \infty]$ with weight e^{-x} produces

$$\widehat{\phi_0} = \frac{1}{\langle 1|e^{-x}|1\rangle} = \frac{1}{1} = 1, \qquad \phi_1 = x - \widehat{\phi_0}\langle\widehat{\phi_0}|e^{-x}|x\rangle = x - 1 = \widehat{\phi}_1,$$
$$\phi_2 = x^2 - \widehat{\phi_0}\langle\widehat{\phi}_0|e^{-x}|x^2\rangle - \widehat{\phi_1}\langle\widehat{\phi}_1|e^{-x}|x^2\rangle = x^2 - 1\cdot 2 - (x-1)\cdot 4 = x^2 - 4x + 2,$$
$$\widehat{\phi}_2 = \frac{\phi_2}{\langle\phi_2|e^{-x}|\phi_2\rangle} = \frac{x^2 - 4x + 2}{2}$$

Comparing these to the recursion in exercise 17.9, namely $L_{n+1} - (2n+1-x)L_n + n^2 L_{n-1}$, we get

$$\widehat{\phi}_{n+1} = \frac{(-1)^{n+1}}{(n+1)!}[(2n+1-x)\widehat{\phi}_n - n^2\widehat{\phi}_{n-1}], \quad n = 1, 2, \ldots$$

for the orthonormal polynomial, which is $\widehat{\phi_3} = (x^3 - 9x^2 + 18x - 6)/6$ in the $n = 2$ case.

17.11. (a) The operator $L = \frac{d}{dx} + x$ acting on the given space is *not* Hermitian:

$$\begin{aligned}\langle f|Lg\rangle &= \int_{-\infty}^{\infty} f^*(x)g'(x)dx + \int_{-\infty}^{\infty} xf^*(x)g(x)dx \\ &= \underbrace{f^*(x)g(x)|_{-\infty}^{\infty}}_{0} + \int_{-\infty}^{\infty} g(x)(-f'^*(x) + xf^*(x))dx \neq \langle Lf|g\rangle.\end{aligned}$$

(b) $L = -i\frac{d}{dx} + x^2$ *is* Hermitian on this space:

$$\langle f|Lg\rangle = \int_{-\infty}^{\infty} f^*(x)(-ig'(x) + x^2 g(x))dx = \underbrace{\int_{-\infty}^{\infty} -if^*(x)g'(x)dx}_{I} + \int_{-\infty}^{\infty} f^*(x)x^2 g(x))dx$$

$$I = -i\underbrace{f'^*(x)g(x)\big|_{-\infty}^{\infty}}_{0} + i\int_{-\infty}^{\infty} f'^*(x)g(x)dx = \int_{-\infty}^{\infty} \left(-if'(x)\right)^* g(x)dx$$

so $\langle f|Lg\rangle = \int_{-\infty}^{\infty} \left(-if'(x) + x^2 f(x)\right)^* g(x)dx = \langle Lf|g\rangle$.

(c) $L = ix\frac{d}{dx}$ is *not* Hermitian:

$$\begin{aligned}\langle f|Lg\rangle &= \int_{-\infty}^{\infty} f^*(x)(ixg'(x))dx = \underbrace{ixf^*(x)g(x)|_{-\infty}^{\infty}}_{0} - i\int_{-\infty}^{\infty}(f^*(x) + xf'^*(x))g(x)dx \\ &= \int_{-\infty}^{\infty} \left(ixf'(x) + xf(x)\right)^* g(x)dx \neq \langle Lf|g\rangle\end{aligned}$$

(d) $L = i\frac{d^3}{dx^3}$ *is* Hermitian:

$$\begin{aligned}\langle f|Lg\rangle &= \int_{-\infty}^{\infty} f^*(x)(ig'''(x))dx = \underbrace{if^*(x)g''(x)\big|_{-\infty}^{\infty}}_{0} - i\int_{-\infty}^{\infty} f'^*(x)g''(x)dx \\ &= -i\underbrace{f'^*(x)g'(x)\big|_{-\infty}^{\infty}}_{0} + i\int_{-\infty}^{\infty} f''^*(x)g'(x)dx \\ &= i\underbrace{f''^*(x)g(x)\big|_{-\infty}^{\infty}}_{0} + \int_{-\infty}^{\infty}(if'''(x))^* g(x)dx = \langle Lf|g\rangle\end{aligned}$$

In each of the above cases, the boundary terms vanish because we assume that the functions to which L applies decay sufficiently fast, and that their derivatives decay sufficiently fast. Precision on this issue is a more advanced topic. See chapter VIII of M. Reed & B. Simon, Methods of Modern Mathematical Physics: *I. Functional Analysis*, Revised and Enlarged Edition, Academic Press 1980.

Though the above calculations suffice, in proving that an operator L is *not* Hermitian it is probably best to find a particular pair of functions f, g in the domain of L and show $\langle f|Lg\rangle \neq \langle Lf|g\rangle$.