Selected Solutions to Assignment # 5

20.18 Suppose that f has a simple zero at $z_0 \in \mathbb{C}$. Then there exists a function $g(z)$, which is analytic and nonzero on a small neighborhood C about z_0, such that $f(z) = g(z)(z - z_0)$. Note that $f'(z) = g'(z) + (z - z_0)g(z)$, so $f'(z_0) = g(z_0)$. Then

$$\text{Res}\left(\frac{1}{f(z)}, z_0\right) = \frac{1}{2\pi i}\oint_C \frac{dz}{f(z)} = \frac{1}{2\pi i}\oint_C \frac{dz}{(z - z_0)g(z)} = \frac{1}{g(z_0)} = \frac{1}{f'(z_0)}.$$

To compute the integral, take the contour C to be the unit circle. Then taking $\sin\theta = (z - z^{-1})/2i$,

$$\int_{-\pi}^{\pi} \frac{\sin\theta d\theta}{a - \sin\theta} = \oint_C \frac{z^2 - 1 \quad dz}{iz(2iaz - z^2 + 1)} = \oint_C \frac{dz}{f(z)}$$

where $f(z) = \frac{iz(2iaz - z^2+1)}{z^2-1}$. This function has a simple zeros in C at $z_0 = 0$ and $z_1 = i(a - \sqrt{a^2 - 1})$. The derivative of f is $-i + 4az/(z^2 - 1)^2$, giving $f'(0) = -i$ and $f'(z_1) = -ia/\sqrt{a^2 - 1}$. The result from the first part gives

$$\int_{-\pi}^{\pi} \frac{\sin\theta d\theta}{a - \sin\theta} = \frac{2\pi i}{f'(z_0) + f'(z_1)} = 2\pi\left(\frac{a}{\sqrt{a^2 - 1}} - 1\right) \text{ after some nasty algebra.}$$

20.19 The equation of an ellipse in polar coordinates with one focus at the origin is $r(\theta) = l(1 + \varepsilon\cos\theta)^{-1}$. Let C be the unit circle. The area of this ellipse is

$$A = \frac{1}{2}\int_0^{2\pi} r^2(\theta)d\theta = \frac{1}{2}\int_0^{2\pi}\left(\frac{l}{1 - \varepsilon\cos\theta}\right)^2 d\theta = \frac{l^2}{2}\oint_C \frac{dz}{iz[1 - \frac{\varepsilon}{4}(z + z^{-1})]^2}dz$$

$$= \frac{2l^2}{i}\oint_C \frac{z\ dz}{(2z - \varepsilon z^2 - \varepsilon)^2}$$

The integrand has poles of order two at $z_{1,2} = \varepsilon^{-1} \pm \sqrt{\varepsilon^{-2} - 1}$. Only $z_1 = \varepsilon^{-1} - \sqrt{\varepsilon^{-2} - 1}$ is in C. So

$$A = \frac{2l^2}{i}\oint_C \frac{z\ dz}{(z - z_1)^2(z - z_2)^2} = 2\pi i\frac{2l^2}{i}\frac{d}{dz}\left[\frac{z}{(z - z_2)^2}\right]_{z=z_1}$$

$$= 4l^2\pi\left[\frac{-z - z_1}{(z - z_1)^3}\right]_{z=z_0} = \frac{\pi\varepsilon l^2}{(1 - \varepsilon^2)^{3/2}}$$

20.24 The function $f(z) = (1 - z^n)^{-1} = (e^{2\pi i} - z^n)^{-1}$ has singularities at $z_0 = e^{i\pi/n}$ since $e^{in\theta} = e^{i\pi}$ when $\theta = \pi/n$. The residue of f at z_0 is

$$\lim_{z\to z_0}\left[(z - z_0)\frac{1}{e^{2\pi i} + z^n}\right] \overset{\text{L'hop}}{=} \lim_{z\to z_0}\frac{1}{nz^{n-1}} = \frac{1}{ne^{i(n-1)\pi/n}} = \frac{-e^{i\pi/n}}{n}.$$

The wedge shaped contour C of angle $2\pi/n$ with one side along the real axis can be defined as the limit of contours C_R which extends along the real axis to $x = R$. The contour integral of the function on C_R can be computed as follows, with the changes of variables $t \to Re^{i\theta}$ and $t \to re^{2\pi i/n}$ in the second and third integrals, respectively

$$\oint_{C_R} \frac{dz}{1+z^n} = \int_0^R \frac{dt}{1+t^n} + \int_0^{2\pi/n} \frac{iRe^{i\theta}d\theta}{1+R^n e^{in\theta}} + \int_R^0 \frac{e^{2\pi i/n}dt}{1+t^n e^{2\pi i/n}} = I_1 + I_2 + I_3.$$

The left hand side is equal to $2\pi i \cdot (-e^{i\pi/n}/n)$ by the Residue Theorem. Note that since I_3 is simply I_1 rotated through an angle of $2\pi i/n$ but is oriented in the opposite direction, $I_3 = -e^{2\pi i/n} I_1$. The value of I_2 vanishes as $R \to \infty$ through application of the Jordan Lemma to the function $g(\theta) = R/(1 + R^n e^{in\theta})$ and a trivial change of variables $w = z^{n/2}$ to deal with the angle of C_R. This stretches out the wedge to fill the upper half plane. Note that this moves the poles at $z = e^{i\pi/n}$, in the center of the wedge, to $w = e^{i\pi/2}$ on the imaginary axis. This leaves

$$\lim_{R\to\infty} I_1 = \int_0^\infty \frac{dt}{1+t^n} = \frac{1}{1-e^{2\pi i/n}} \cdot \frac{-2\pi i e^{i\pi/n}}{n} = \frac{\pi}{n} \cdot \frac{-2i}{e^{-i\pi/n} - e^{i\pi/n}} = \frac{\pi}{n} \csc \frac{\pi}{n}$$

20.32 Note that the transformed function

$$F(s) = \frac{e^{-s} - 1 + s}{s^2} = \frac{\left(1 - s + \frac{s^2}{2} - \frac{s^3}{3!} + \cdots\right) - (1-s)}{s^2} = \frac{1}{2} - \frac{s}{3!} + \frac{s^2}{4!} - \cdots$$

has no pole at $s = 0$. Take $F_1(s) = e^{-s}/s^2$ and $F_2(s) = (s-1)/s^2$. Recall that the inverse transform is linear, so $\mathcal{L}^{-1}[F] = \mathcal{L}^{-1}[F_1] + \mathcal{L}^{-1}[F_2]$.

For $x < 0$, take the Bromwich contour C_1 toward the right side of the plane. Since there are no poles in this region, the Cauchy theorem gives $\mathcal{L}^{-1}[F] = f(x) = 0$ for $x < 0$. For $x \geq 1$ using the contour C_2 toward the left half plane, $L^{-1}[F]$ is again zero since the exponent of $e^{s(x-1)}$ is negative. When $x \in (0, t)$, $f(x)$ can be computed by

$$\frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \frac{e^{s(x-1)}}{s^2} + \frac{e^{sx}(s-1)}{s^2} ds = \frac{1}{2\pi i} \left(\oint_{C_1} \frac{e^{s(x-1)}}{s^2} ds + \oint_{C_2} \frac{e^{sx}(s-1)}{s^2} ds \right) = R_1 + R_2$$

. Then $R_1 = 0$ since $s = 0$ is not included in C_1 for $\lambda > 0$ and $R_2 = \frac{1}{2\pi i}\text{Res}(e^{sx}(s-1)/s^2) = x(s-1)e^{sx} + e^{sx} = -x + 1 = 1 - x$. Combining these yeilds

$$f(x) = \begin{cases} 1 - x & x \in (0,1) \\ 0 & x \notin (0,1). \end{cases}$$