Selected Solutions to Assignment #4

Exercises H, I, and K were graded at five points each, and 20.13 at 10 points, for a total of 25 points.

Exercise H. Let C is a positively-oriented simple closed curve. Then

$$\begin{aligned}\oint_C z^*\,dz &= \oint_C (x-iy)(dx+i\,dy) = \oint_C (x\,dx+y\,dy)+i(-y\,dx+x\,dy)\\ &= \iint_{\partial C}\left(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right)+i\left(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}\right)\,dA \quad \text{(by Green's Theorem)}\\ &= \iint_{\partial C} 0+2i\,dA = 2i\,A,\end{aligned}$$

which gives the desired result.

Exercise I. Let C be any simple closed positively oriented curve in $\mathbb{C}$. Define

$$g(w) = \oint_C \frac{z^3+2z}{(z-w)^3}\,dz.$$

When w is outside the region enclosed by C, note that $h(z) = (z^3+2z)/(z-w)^3$ is analytic in C as well as on the boundary of C. By the Cauchy Theorem, $\oint_C h(z)\,dz = 0$. This means $g(w) = 0$. For w inside of C, the value of g is found using the formula

$$\frac{d^n}{dz^n}f(w) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-w)^{n+1}}dz$$

with $n = 2$ and $f(z) = z^3+2z$. The result is

$$g(w) = \oint_C \frac{z^3+2z}{(z-w)^3}\,dz = \frac{2\pi i}{2!}\frac{d^2}{dz^2}\left(z^3+2z\right)_{z=w} = 6\pi i w$$

Exercise K. Let C be the unit circle $z = e^{i\theta}$, $-\pi \le \theta \le pi$ and suppose $a \in \mathbb{R}$ with $f(z) = e^{az}$. Note that f is an entire function, so the Cauchy Integral Formula can be applied with $z_0 = 0$:

$$\oint_C \frac{e^{az}}{z-z_0}\,dz = 2\pi i e^{az_0} = 2\pi i.$$

Changing the variable of integration using the substitutions $z = e^{i\theta}$ and $dz = ie^{i\theta}d\theta$,

$$\begin{aligned}\oint_C \frac{e^{az}}{z}\,dz &= \int_{-\pi}^{\pi} \frac{e^{a\cdot e^{i\theta}}}{e^{i\theta}}(ie^{i\theta}d\theta) = i\int_{-\pi}^{\pi} e^{ae^{i\theta}}\,d\theta = i\int_{-\pi}^{\pi} e^{a\cdot \mathrm{cis}\theta}d\theta\\ &= i\int_{-\pi}^{\pi} e^{a\cos\theta}e^{ia\sin\theta}d\theta = i\int_{-\pi}^{\pi} e^{a\cos\theta}(\cos(a\sin\theta)+i\sin(a\sin\theta))d\theta\\ &= i\int_{-\pi}^{\pi} e^{a\cos\theta}(\cos(a\sin\theta))d\theta - \int_{-\pi}^{\pi} e^{a\cos\theta}(\sin(a\sin\theta))d\theta\end{aligned}$$

The second term is zero since it is the integral of an odd function on a symmetric interval. This, along with the result from the first part, gives

$$i\int_{-\pi}^{\pi} e^{a\cos\theta}(\cos(a\sin\theta))\,d\theta = 2\pi i$$

or, by the even symmetry of the integrand

$$\int_0^\pi e^{a\cos\theta}(\cos(a\sin\theta))\,d\theta = \pi.$$

20.13. Consider the function $f(z) = e^{-az^2}$ where $a > 0$. This can be written as the composition of entire functions e^{-z} and az^2, so it is entire. By the Cauchy Theorem, $\oint_C f(z)dz = 0$ for any simple closed curve in the plane. Let C be the circular sector of radius b and angle $pi/4$ in the first quadrant with one side aligned with the real axis. Then the contour can be parameterized in three different paths [thanks Rob, these make it much easier]:

$$C_1 : \gamma_1(t) = bt, t \in [0,1] \qquad C_2 : \gamma_2(\theta) = be^{i\theta}, \theta \in [0,\pi/4] \qquad C_3 : \gamma_3(t) = (1-t)be^{i\pi/4}, t \in [0,1]$$

The integral of f on the first curve is simply $\int_{C_1} f(z)dz = \int_0^b e^{-ax^2}dx$ as the path is on the real axis. The value of this integral as $b \to \infty$ is $\sqrt{\pi/(4a)}$ which can be derived from exercise L(b). Along the path C_2, $z = be^{i\theta}$ and $dz = bie^{i\theta}d\theta$

$$\begin{aligned}\int_{C_2} f(z)dz &= \int_0^{\pi/4} \exp(-a(be^{i\theta})^2)\cdot(bie^{i\theta}d\theta) = bi\int_0^{\pi/4}\exp(-ab^2e^{2i\theta})e^{i\theta}d\theta \\ &= bi\int_0^{\pi/4}\exp\left[-ab^2\cos 2\theta + i(-ab^2\sin 2\theta + \theta)\right]d\theta\end{aligned}$$

The sum in this exponential can be written as multiplication at the base. Note that the imaginary parts in this expression have unit magnitude and the real part is nonnegative, hence

$$\left|\int_{C_2} f(z)dz\right| \le b\int_0^{\pi/4}\left|e^{-ab^2\cos 2\theta}\right|\left|e^{-iab^2\sin 2\theta}\right|\left|e^{i\theta}\right|d\theta = b\int_0^{\pi/4} e^{-ab^2\cos 2\theta}d\theta$$

To demonstrate that this integral vanishes as $b \to \infty$, consider that $\exp(-ab^2\cos 2\theta) \le \exp(-ab^2)\exp(4ab^2\theta/\pi)$ on the interval $\theta \in [0,\pi/4]$. Consequently,

$$\int_0^{\pi/4}\exp(-ab^2\cos 2\theta)d\theta \le e^{-ab^2}\int_0^{\pi/4}\exp(4ab^2\theta/\pi)d\theta = \frac{1-e^{-ab^2}}{ab^2(4/\pi)}$$

and it follows that $\lim_{b\to\infty}\int_{C_2} f(z)dz = 0$ since it is dominated by an integral which also vanishes in the same limit.

Finally, the third contour integral is computed using the parameterization of γ_3 and the change of variables $z = (1-t)be^{i\pi/4}$ and $dz = -be^{i\pi/4}dt$. This yields

$$\begin{aligned}\int_{C_3} f(z)dz &= \int_0^1 \exp\left(-a[(1-t)be^{i\pi/4}]^2\right)\left(-be^{i\pi/4}\right) \\ &= -be^{i\pi/4}\int_0^1 \exp\left(-iab^2(1-t)^2\right) \\ &= \frac{1+i}{\sqrt{2}}\int_b^0 \cos(ax^2) - \sin(ax^2)dx \qquad \text{with} \quad \begin{matrix} x = (1-t)b \\ dx = -bdt \end{matrix} \\ &= \frac{-1}{\sqrt{2}}(C_b + S_b) + \frac{i}{\sqrt{2}}(S_b - C_b)\end{aligned}$$

where $C_b = \int_0^b \cos(ax^2)dx$ and similarly for S_b. Also, define $C = \lim_{b\to\infty} C_b$. Pulling all the paths together and letting $b \to \infty$,

$$0 = \lim_{b\to\infty} \left(\int_{C_1} + \int_{C_2} + \int_{C_3} \right) f(z)dz = \frac{1}{2}\sqrt{\frac{\pi}{a}} + 0 - \frac{1}{\sqrt{2}}(C+S) - \frac{i}{\sqrt{2}}(C-S)$$

The imaginary parts must be zero, so $C = S$. It follows that $(S+C)/\sqrt{2} = \sqrt{2}C = \sqrt{\pi/(4a)}$, or

$$C = \int_0^\infty \cos(ax^2)\,dx = \sqrt{\frac{\pi}{8a}}.$$