Selected Solutions to Assignment #3

I graded exercises E, F, G, and 20.12 at five points each for a total of 20 points.

Exercise F. If $f(z) = (az+b)/(cz+d)$ we immediately have the equations at left, which can be rewritten as those at right:

$$1 = \frac{a \cdot 2 + b}{c \cdot 2 + d} \qquad\qquad 2a + b - 2c - d = 0$$

$$i = \frac{a \cdot i + b}{c \cdot i + d} \qquad\qquad ia + b + c - id = 0$$

$$-1 = \frac{a \cdot (-2) + b}{c \cdot (-2) + d} \qquad\qquad -2a + b - 2c + d = 0$$

The equations at right are a system of linear equations with many solutions. Among the solutions is $a = 0, b = 0, c = 0, d = 0$, and that is not useful to us. If we *choose* a value, say $b = 1$, we get a system of equations with only one solution:

$$\begin{aligned} 2a - 2c - d &= -1 \\ ia + c - id &= -1 \\ -2a - 2c + d &= -1 \end{aligned}$$

This system can be solved by hand with ease, but just to suggest how one might do it automatically, here is the Matlab:

```
>> A=[2 -2 -1; i 1 -i; -2 -2 1];  b=[-1 -1 -1]';  A\b
ans =
        0 -         1.5i
      0.5
        0 -           3i
```

One gets $a = -(3/2)i$, $c = 1/2$, and $d = -3i$, so

$$f(z) = \frac{-(3/2)iz + 1}{(1/2)z - 3i} = \frac{3z + 2i}{iz + 6}.$$

It is easy to check that this is correct.

The following corrects the earlier incorrect version: Recall that linear fractional transformations send lines to lines or circles, so the answer must be a line or a circle. In this case it is easiest (at this late stage) to avoid a mass of algebra and use Matlab to show in the circle which is produced if one applies the linear fractional transformation to (part of!) the line $y = x$ in the input plane, that is, to the line $z(t) = t + it$:

```
>> t=-10:.01:10; w=(3*(t+i*t)+2*i)./(i*(t+i*t)+6);
>> plot(real(w),imag(w)), axis equal, grid on
```

Figure 1 results. It turns out that this circle has center $(5/3, -4/3)$ and radius $5\sqrt{2}/3 \approx 2.357$, which is made credible, but not proven, by the figure.

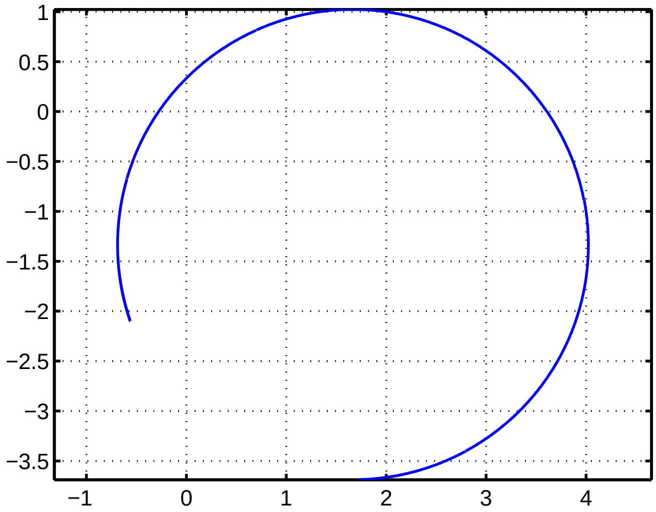

FIGURE 1. MATLAB picture of part of the image of the line $y = x$ under the linear fractional transformation in **Exercise F**.

Exercise G. **(a)** If

$$f(z) = \frac{az+b}{cz+d} \qquad \text{and} \qquad g(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$$

then

$$f(g(z)) = \frac{a\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)+b}{c\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)+d} = \frac{(a\alpha+b\gamma)z+(a\beta+b\delta)}{(c\alpha+d\gamma)z+(c\beta+d\delta)}$$

(b) If $T(z) = (az+b)/(cz+d)$ then the following statements are equivalent from part **(a)**:

$$T^{-1}(z) = T(z) \iff z = T(T(z)) \iff z = \frac{(a^2+bc)z+(ab+bd)}{(ca+dc)z+(cb+d^2)}$$

The last form can be written out on one line and simplified:

$$\begin{aligned} & (ca+dc)z^2+(cb+d^2)z = (a^2+bc)z+(ab+bd) \\ \iff \quad & (a+d)cz^2+(d^2-a^2)z-(a+d)b = 0 \\ \iff \quad & (a+d)\left[cz^2+(d-a)z-b\right] = 0 \end{aligned}$$

These equations must be true *for all* z if were are to have $T^{-1}(z) = T(z)$. The last of these equations therefore says that

$$\textit{either} \qquad a+d=0 \qquad \textit{or} \qquad cz^2+(d-a)z-b=0 \text{ for all } z.$$

In the latter case, where we have a quadratic polynomial that is zero, the coefficients must be zero: $c = 0$, $d-a = 0$, $b = 0$.

We conclude that the only forms of self-inverse linear fractional transformations are:

$$T(z) = \frac{az+b}{cz-a} \qquad \text{or} \qquad T(z) = z.$$