Assignment #1

Due *Thursday, 26 January 2006.*

I. Though we stopped in the middle of differential equations and introductory-functional-analysis matters at the end of last semester, I want to start with something relatively "clean." We will do complex analysis, chapter 20. This will explain much we have already used (e.g. convergence of power series, behavior of the complex exponential) but will also give us new tools we can use (e.g. conformal maps, contour integrals). You may want some other source on complex analysis. The text for Math 422, Churchill and Brown's *Complex Variables and Applications*, fits the bill nicely. I will not actually use this alternate in any concrete way. I just think that a good book-length treatment of the subject both fleshes it out and presents it more clearly than our text.

II. Read sections 20.1, 20.2, 20.3, 20.4, and 20.6 of Riley, Hobson, & Bence. Recall the notation: if $z = x + iy$ then $z^* = x - iy$; other books write "$\bar{z} = x - iy$." Note the text uses the archaic language "Argand diagram" for the complex plane.

III. Do exercises:

Exercise A. Express $\operatorname{Re} z^{-2}$ in terms of x and y assuming $z = x + iy$.

Exercise B. (a) Along the lines of the example on top of page 712, show using the definition (20.1) that $g(z) = (z+1)/z$ is differentiable when $z = 3 + 2i$; compute $g'(3 + 2i)$.

(b) Confirm that

$$u(x, y) = \operatorname{Re} g(z) = 1 + \frac{x}{x^2 + y^2}, \qquad v(x, y) = \operatorname{Im} g(z) = -\frac{y}{x^2 + y^2}.$$

(That is, confirm that I have given the correct formula for the real and imaginary parts of $g(z)$). Use the Taylor's approximation in two variables

$$u(x, y) = u(x_0, y_0) + \frac{\partial u}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial u}{\partial y}(x_0, y_0)(y - y_0),$$

and the corresponding formula for $v(x, y)$, to give linear approximations to $u(x, y)$, $v(x, y)$ at $(x_0, y_0) = (3, 2)$.

(c) Relate the computations in parts (a) and (b). That is, write a precise paragraph.

Exercise C. Sketch (i.e. shade) the regions in the complex plane:

$$\text{(a)}\ \operatorname{Re} z^2 < 1, \qquad \text{(b)}\ |\operatorname{Im} \exp(z)| < 1.$$

(This exercise is to get you to think about the behavior of the familiar functions when the inputs and outputs are complex. You can check your work by machine, I suppose, but you will get nothing out of it if you don't do it by hand.)

Exercise D. Explain the sentence "The Cauchy-Riemann equations give conditions under which a function $f(x, y) = u(x, y) + iv(x, y)$ of two real variables can be written as a function of a single complex variable $f(z)$." Now give equations which must apply to u, v if the function $f(x, y) = u(x, y) + iv(x, y)$ of two real variables is to be written as $f(z^*)$.

IV. Do exercises from Riley, Hobson, & Bence:

20.1, 20.3.