ODE problems as "matrix" problems: A first try

Consider the ODE problem

$$y'' + 4y = x, \qquad y(0) = 0, \qquad y'(0) = 0. \tag{1}$$

Can we solve it by classical means? Absolutely! Here goes: The characteristic (polynomial) equation of the associated homogeneous ODE $y_h'' + 4y_h = 0$ is $m^2 + 4 = 0$ with roots $m_{1,2} = \pm 2i$. Thus $y_h = c_1 \sin 2x + c_2 \cos 2x$ for some constants [section 15.1 of Riley, Hobson, & Bence]. Next, a particular solution exists of the form $y_p(x) = Ax + B$ [subsection 15.1.2 of RHB] because x is not a solution of the homogeneous problem. Substituting into the ODE in (1) we have $0 + 4(Ax + B) = x$ so $B = 0$ and $A = 1/4$. Thus our general solution is $y(x) = c_1 \sin 2x + c_2 \cos 2x + x/4$. The initial conditions imply $0 = c_2$ and $0 = 2c_1 + 1/4$. Thus the solution to (1) is

$$y(x) = -\frac{1}{8}\sin 2x + \frac{x}{4}.$$

Now, on the other hand, a major point in class was that there is a linear object in (1), an "operator", which we must *invert* to solve the problem. I proved the following in class:

Lemma.

$$L = \frac{d^2}{dx^2} + 4$$

is linear. That is, for any pair of real numbers a, b and any two functions $f = f(x)$ and $g = g(x)$,

$$L(af + bg) = aL(f) + bL(g).$$

Thus problem (1) can be written

$$Ly = x.$$

We seek to invert L and write

$$\text{“}y = L^{-1}x.\text{”}$$

The classical method above apparently accomplishes this inversion of L! The initial values were *crucially* involved in the inversion, because only with those initial values do we have a *unique* solution.

In linear algebra this uniqueness "goes with" a matrix inverse. That is, we have the statement: If b is an $n \times 1$ vector and A a nonsingular $n \times n$ matrix then the problem $Av = b$ has a unique solution v and we can compute it $v = A^{-1}b$.

Now, if one has a linear object (a "map", or "operator", or whatever ...) which one wants to understand then one usually looks at its representation in a basis. That is, one wants to see it as a *matrix.*

For problem (1), the issue in producing a matrix-vector representation is

what is the vector space on which L acts?

Clearly it is a vector space of functions. And it seems that L will only make sense if it is a space of *twice-differentiable* functions. Less clear, however, is what kinds of functions to include into the function space, in terms of "size".

The bases[1] of functions we already know about include: (*i*) $\{x^n\}$ for $n = 0, 1, 2, \dots,$ for Taylor series, (*ii*) $\{\cos nx\}$ (and other sinusoids) with discrete frequencies n, for Fourier series, (*iii*)

[1] I.e. the plural of "basis". Pronounced *base-ees.*

$\{\exp(i\omega x)\}$ with continuous frequencies ω, for the Fourier transform, and (iv) $\{\exp(-sx)\}$ with continuous "frequencies" s—decay rates, actually—for the Laplace transform.

In each case, these bases don't allow the representation of arbitrary functions, and, in different ways in each case, restrict the "size" of the possible functions one can represent. Fourier series methods, for example, require we use periodic functions, which are bounded if they are continuous. Fourier transform methods require functions which have finite integral (i.e. $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$).

Let's try the two *discrete* bases mentioned above.

Matrix representation I. Consider the Taylor series basis

$$\{1, x, x^2, x^3, \dots\} = \{x^n\}_{n=0}^{\infty}$$

This is, at the minimum, a basis for the space of all polynomials of all degree. But we also regard e^x as a function in the span of this basis:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots$$

Now, L acts in this basis in an easily calculated way:

$$L(x^n) = \frac{d^2}{dx^2}(x^n) + 4(x^n) = n(n-1)x^{n-2} + 4x^n. \tag{2}$$

As special cases, $L(x^0) = 4 \cdot 1$ and $L(x^1) = 4 \cdot x$. Thus

$$L = \begin{pmatrix} 4 & 0 & 2\cdot 1 & 0 & \dots \\ 0 & 4 & 0 & 3\cdot 2 & \\ 0 & 0 & 4 & 0 & \ddots \\ 0 & 0 & 0 & 4 & \\ \vdots & & & & \ddots \end{pmatrix}$$

In particular, L has entries "4" on the diagonal and entries $n(n-1)$ on the second superdiagonal. To state this clearly, if a function is represented by a power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

then we think of its coefficients as a column vector. For example,

$$x^2 \sim \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \qquad \text{and} \qquad 1 + 17x - x^3 \sim \begin{bmatrix} 1 \\ 17 \\ 0 \\ -1 \\ \vdots \end{bmatrix}.$$

Then

$$Lf = \begin{pmatrix} 4 & 0 & 2\cdot 1 & 0 & \dots \\ 0 & 4 & 0 & 3\cdot 2 & \\ 0 & 0 & 4 & 0 & \ddots \\ 0 & 0 & 0 & 4 & \\ \vdots & & & & \ddots \end{pmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 2\cdot 1 a_2 + 4a_0 \\ 3\cdot 2 a_3 + 4a_1 \\ 4\cdot 3 a_4 + 4a_2 \\ 5\cdot 4 a_5 + 4a_3 \\ \vdots \end{bmatrix},$$

which is just a rewritten form of computation (2).

Can we solve $Ly = x$ from the original problem (1)? Note that the right hand side is easy to represent, because x is its own Taylor series. We can write

$$\begin{pmatrix} 4 & 0 & 2\cdot 1 & 0 & \dots \\ 0 & 4 & 0 & 3\cdot 2 & \\ 0 & 0 & 4 & 0 & \ddots \\ 0 & 0 & 0 & 4 & \\ \vdots & & & & \ddots \end{pmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}. \tag{3}$$

That is, we seek a Taylor expansion of the solution

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

We are now in familiar territory, or, at least, Math 302 territory. That is, the equations $4a_0 + 2a_2 = 0$, $4a_1 + 3\cdot 2a_3 = 1$, $4a_2 + 4\cdot 3a_4 = 0$, $4a_3 + 5\cdot 4a_5 = 0$, ... are exactly the equations we would solve to use the series expansion methods in section 16.2 of Riley, Hobson, & Bence.

Disturbingly, however, we note a pair of facts about (3):

(1) The matrix on the left side of (3) is upper triangular (pp. 274-275 of RHB) and has nonzero entries on the diagonal. *If* this matrix were of finite size then it would be invertible and there would be a unique solution to the matrix problem.

(2) There is *not* a unique solution to (3), and *we don't believe there should be a unique solution.* In terms of the original problem (1), we have not included the initial values into (3). This fact is reflected in the "series expansion method" above, which would leave a_0, a_1 as unknowns until one used the initial values.

So we must be careful with "infinity-by-infinity matrices," apparently!

Matrix representation II. Consider the real Fourier series basis

$$\{1, \sin x, \cos x, \sin 2x, \cos 2x, \dots\}.$$

Again L acts in this basis in an easily calculated way:

$$L(\sin nx) = \frac{d^2}{dx^2}(\sin nx) + 4(\sin nx) = (-n^2 + 4)\sin nx,$$
$$L(\cos nx) = \frac{d^2}{dx^2}(\cos nx) + 4(\cos nx) = (-n^2 + 4)\cos nx.$$

(Note that the $n = 0$ case is correctly computed: $1 = \cos(0x)$.)

This means L comes out as a *diagonal* matrix! In fact,

$$L = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & -5 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 & \\ & & \vdots & & & & & \ddots \end{pmatrix}$$

(Note that $-0^2 + 4 = 4$, $-1^2 + 4 = 3$, $-2^2 + 4 = 0$, $-3^2 + 4 = -5$, ...)

Thus L is diagonal, but two zeros appear on the diagonal. This is good! As noted above, we do not expect to be able to invert L, literally, because we know that we must "add in" the two initial conditions in order to get a unique solution.

In terms of linear algebra, we see immediately that the rank is two less than infinity, the size of the matrix. To solve problem (1) we can modify the two rows that have zero in the diagonal and then we can invert the modified matrix.

Actually implementing such a scheme is difficult. We need to rewrite the two initial conditions $y(0) = 0$ and $y'(0) = 0$ in the Fourier series representation. In particular, if we have a Fourier series representation of the Dirac δ function we can do it. Suppose[2]

$$\delta(x) = \frac{d_1}{2} + \sum_{n=1}^{\infty} d_{2n}\sin(nx) + d_{2n+1}\cos(nx).$$

I claim it would be reasonable to replace row 4 of L above with the list

$$\begin{pmatrix} d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & \dots \end{pmatrix}.$$

A similar replacement of row 5 with the Fourier coefficients of the *derivative* of the Dirac δ function would also be required.

Unfortunately, the modified matrix L is no longer diagonal!

I need a punchline at this point. Consider an nth order linear ordinary differential equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \tag{4}$$

which corresponds to a linear operator

$$L = \frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_1(x)\frac{d}{dx} + p_0(x).$$

Slogan. *L has* nullity *n. That is,*

the dimension of the space of solutions to $Ly = 0$ *is* n.

In classical terms we say that the homogeneous equation (4) has n linearly-independent solutions $y_1(x), \dots, y_n(x)$ and so its general solution is $y(x) = c_1y_1(x) + \cdots + c_ny_n(x)$.

So one requires n "independent" pieces of data to determine a particular solution to (4). It is well-known that *initial* data, that is, values

$$y_0 = y(x_0), \quad y_0' = y'(x_0), \quad \dots, \quad y_0^{(n-1)} = y^{(n-1)}(x_0),$$

are always "independent" in this sense.[3] It is also well-known that n pieces of *boundary* data *may not be* "independent" in this sense.

Note that this linear operator may or may not have a *useful* matrix representation in the basis you choose! And to get a matrix representation of L you do have to *choose* a basis. But we will see that the abstract concept of a linear operator is very valuable.

The "nullity" of a matrix is defined on page 298 of Riley, Hobson, & Bence. Nullity is the complementary idea to the "rank" of a matrix: the *nullity* of an $n \times n$ matrix A is the number of linearly-independent solutions of the equation $Av = 0$ while the *rank* of an $n \times n$ matrix A is

[2] It is an exercise to find the coefficients d_k; not assigned but also not hard. The indexing scheme is reasonable from the point of view of our "matrix" thinking, by the way, even if it doesn't match the scheme in section 12 of RHBfor Fourier series.

[3] The proof of this statement is to use a (potentially) *nonlinear* tool, namely the general existence and uniqueness theorem for ordinary differential equations.

the number of linearly-independent columns[4] of A. One can go back-and-forth between rank and nullity because

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n.$$

The nullity of a non-singular $n \times n$ matrix is zero. In fact, a square matrix is invertible *if and only if* it has nullity zero. At the other extreme, the nullity of the $n \times n$ *zero* matrix is n.

Exercise B. Consider the scheme for representing Taylor series as column vectors (in the first "matrix representation" scheme above). By directly considering radii of convergence, explain why f below does not really represent a function, while g below does represent a function but $g(2)$ does not make sense:

$$f \sim \begin{bmatrix} 1 \\ 2 \\ 6 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1! \\ 2! \\ 3! \\ \vdots \end{bmatrix} \qquad g \sim \begin{bmatrix} 4 \\ 15 \\ 54 \\ \vdots \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \cdot 3 \\ 6 \cdot 3^2 \\ \vdots \end{bmatrix}$$

Can you think of any other problems with using this "Taylor coefficients as a column vector" representation scheme casually?

Exercise C. The matrix of L in the second representation scheme above is a bit *too* nice. It is indeed diagonal, and, because two of the diagonal entries are zero, we "see immediately that the rank is two less than infinity." So I argue above that adding two initial conditions allows us to "invert the modified matrix." But how representative is this case?

To see some of the problem, find the matrix which represents the operator

$$\tilde{L} = \frac{d^2}{dx^2} + \frac{d}{dx} + 4$$

in the same (Fourier series) basis. Relate the entries of the resulting matrix (*hint*: it is also diagonal) to inversion of the matrix, on the one hand, and the need for initial conditions to make a unique solution to the ODE problem $y'' + y' + 4y = f(x)$, on the other. Relate the fact that the Fourier series must represent a *periodic* function to the matrix representations of L and $\tilde{L}$, respectively.

Exercise D. In fact, the second representation scheme above is much more appropriate for *boundary value problems*. For example, let's suppose that we want to solve $y'' + 4y = x$ on the interval $[0, 1]$ with boundary values $y(0) = 0$ and $y(1) = 0$. Note that every element in the list

$$\beta = \{\sin \pi x, \sin 2\pi x, \sin 3\pi x, \dots\}$$

also has zero values at each end of the interval $[0, 1]$. Supposing β is a basis for the desired function space, find the matrix representation of L. Solve the boundary value problem by also finding the representation of x in the basis β (i.e. compute a Fourier sine series) and *inverting* L to solve "$Ly = x$". Check that you really have a solution. Finally, answer the question: could the coefficient "4" in the formula for L be replaced by any number and the calculation go through? If not, what numbers are problematic?

---

[4]Or rows, interestingly.