Solutions/Answers to Midterm Exam

Closed book, in-class, 1.5 hours, one sheet of notes allowed.

1. *Find the general solution to the ODE $2xy' + 3x + y = 0$. (Find a formula which applies, at least, for $x > 0$.)*

It's linear:

$$y' + \frac{1}{2x}y = -\frac{3}{2}.$$

The integrating factor is

$$\mu(x) = e^{\int (2x)^{-1}\,dx} = e^{(1/2)\ln x} = x^{1/2}.$$

Thus $\left(x^{1/2}y\right)' = -(3/2)x^{1/2}$ so $y(x) = -x + cx^{-1/2}$.

2. *Find the function whose Laplace transform is $e^{-st_0}/(s^2+5)$. (Assume $t_0 > 0$ if needed.)*

One method is to do a convolution given that $\mathcal{L}[\delta(t-t_0)](s) = e^{-st_0}$ and $\mathcal{L}[\sin bt](s) = b/(s^2+b^2)$.

An alternate method, done here, is to recall:

$$\mathcal{L}\left[f(t-t_0)H(t-t_0)\right](s) = \int_{t_0}^{\infty} f(t-t_0)e^{-st}\,dt = e^{-st_0}\int_0^{\infty} f(u)e^{-su}\,du = e^{-st_0}\bar{f}(s).$$

Thus we need $f(t)$ so that $\bar{f}(s) = 1/(s^2+5)$. But then $f(t) = (\sqrt{5})^{-1}\sin(\sqrt{5}t)$. The result is that if

$$g(t) = \frac{1}{\sqrt{5}}\sin\left(\sqrt{5}(t-t_0)\right)H(t-t_0)$$

then $\bar{g}(s) = e^{-st_0}/(s^2+5)$ as given.

3. **(a)** *Find the general solution to the ODE*

$$\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} - 6\frac{dy}{dt} = 0.$$

The characteristic polynomial equation is

$$\lambda^3 - \lambda^2 - 6\lambda = 0 \qquad \text{so} \qquad y(t) = c_1 + c_2e^{3t} + c_3e^{-2t}.$$

(b) *Solve the initial value problem $\dddot{x} - \ddot{x} - 6\dot{x} = t$, $x(0) = 0$, $\dot{x}(0) = 1/36$, $\ddot{x}(0) = -1/6$. You need not redo any work done in part **(a)** above.*

First we need a particular solution and *then* we will satisfy the initial conditions. Trying the form "$x_p(t) = At + B$" is not adequate because $x(t) = B$ solves the homogeneous version of the equation; see part **(a)**. So we multiply by t and try $x_p(t) = At^2 + Bt$. From substitution in the nonhomogeneous ODE we get $A = -1/12$ and $B = 1/36$ so

$$x(t) = c_1 + c_2e^{3t} + c_3e^{-2t} - \frac{1}{12}t^2 + \frac{1}{36}t$$

as the general solution.

The initial data gives the equations

$$\begin{array}{ccccccc} c_1 & + & c_2 & + & c_3 & = & 0 \\ & & 3c_2 & - & 2c_3 & = & 0 \\ & & 9c_2 & + & 4c_3 & = & 0. \end{array}$$

This is a nonsingular system so $c_1 = c_2 = c_3 = 0$. Thus

$$x(t) = -\frac{1}{12}t^2 + \frac{1}{36}t.$$

4. **(a)** *Find a (real, i.e. classical) Fourier series for*

$$f(t) = \begin{cases} -1, & -1 < t < 0, \\ +1, & 0 < t < 1, \end{cases}$$

which has period 2.

The function is *odd*, so the classical Fourier series is a sine series with period $L = 2$. The coefficients are

$$b_r = \frac{2}{2}\int_{-1}^{1} f(t)\sin(2\pi rt/2)\,dt = 2\int_{0}^{1} 1\cdot\sin(\pi rt)\,dt = 2\frac{-\cos(\pi rt)}{\pi r}\Big|_0^1$$
$$= \frac{2}{\pi r}(1-(-1)^r) = \frac{2}{\pi r}\begin{cases}0, & r \text{ even},\\ 2, & r \text{ odd}.\end{cases}$$

Thus the series is

$$f(t) = \sum_{k=0}^{\infty}\frac{4}{\pi(2k+1)}\sin(\pi(2k+1)t).$$

(b) *Use Parseval's theorem on the answer to* ***(a)*** *to sum the series* $\sum_{n=0}^{\infty}(2n+1)^{-2}$.

For a classical Fourier series with $a_r = 0$ for all r and period 2, Parseval's theorem says

$$\frac{1}{2}\int_{-1}^{1}|f(t)|^2\,dt = \frac{1}{2}\sum_{r=1}^{\infty}|b_r|^2.$$

In our case $|f(t)|^2 = 1$, so the quantity on the left is just 1. Writing the sum on the right as over only the odd numbers,

$$1 = \frac{1}{2}\sum_{k=0}^{\infty}\frac{4^2}{\pi^2(2k+1)^2}.$$

Factoring the constant on the right and changing the index to n we get

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8},$$

which is thus true, but really not obvious.

[*However, using* MATLAB *to sum the first 5,000,000 terms one gets* 1.23370050, *in less than one second, while* $\pi^2/8 = 1.23370055$ *to the same number of digits. This represents a relative error of less than one part in* 10^7. *Thus with modern brute force we have no trouble believing such results.*

By the way, what is your opinion*: does the sum of the inverse squares, or inverse odd squares as here, converge quickly or slowly?*]

5. *Use de Moivre's theorem to prove the standard trigonometric identity* $\sin 2\theta = 2\sin\theta\cos\theta$.

de Moivre's theorem notes $e^{in\theta} = (e^{i\theta})^n$. In the $n = 2$ case we find

$$(e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta).$$

On the other hand, $e^{i2\theta} = \cos 2\theta + i\sin 2\theta$. Equating imaginary parts in these two expressions, we have proved the trig. identity.

6. *Show, using the definition of the Fourier transform, that* $\mathcal{F}[df/dt] = i\omega\tilde{f}(\omega)$, *where* $\tilde{f} = \mathcal{F}[f]$. *Assume, where needed, that* $f(t)\to 0$ *as* $|t|\to\infty$.

$$\mathcal{F}\left[\frac{df}{dt}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{df}{dt}(t)e^{-i\omega t}\,dt = \frac{1}{\sqrt{2\pi}}\left\{\left[f(t)e^{-i\omega t}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)(-i\omega)e^{-i\omega t}\,dt\right\}$$
$$= i\omega\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = i\omega\tilde{f}(\omega).$$

The term evaluated at $\pm\infty$ is zero by the allowed assumption.

7. *Is the following first order ODE exact?:* $xy' + 3x + y = 0$

YES. Rewriting the equation in differential form we have

$$x\,dy + (3x+y)\,dx = 0.$$

Exactness requires that the term in front of "dy" is the y derivative of a function of two variables and the term in front of "dx" is the x derivative of that same function. The equality of the mixed partials of the unknown function of two variables is the criterion for exactness:

$$\frac{\partial}{\partial x}(x) \stackrel{?}{=} \frac{\partial}{\partial y}(3x+y).$$

But we see that this criterion is satisfied because each side simplifies to 1.

8. *Differentiate the Fourier cosine series, which is valid on* $[-1,1]$,

$$|t| = \frac{1}{2} - \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi^2}\cos((2k+1)\pi t),$$

to get a new Fourier series. What is the value of the resulting series at $t=1$*?*

THIS PROBLEM CONTAINS A TYPO AND I DID NOT GRADE IT!

The given Fourier cosine series *should* have a square on the "$(2k+1)$" in the denominator:

$$|t| = \frac{1}{2} - \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2}\cos((2k+1)\pi t).$$

With this change it is correct, and differentiable at all points *except* $t=-1,0,1$. The derivative is computable term-by-term:

$$\frac{d}{dt}|t| = 0 - \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2}(-(2k+1)\pi)\sin((2k+1)\pi t) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi}\sin((2k+1)\pi t).$$

On the other hand,

$$\frac{d}{dt}|t| = \begin{cases} -1, & -1<t<0, \\ +1, & 0<t<1. \end{cases}$$

Thus we have solved **4(a)** as well. The value of $\frac{d}{dt}|t|$ is not defined at $t=1$, as noted, but the series has value zero because that is halfway in the jump.

9. *Find a function of t whose Laplace transform is* $(s^3-3s^2-4s+12)^{-1}$. *[HINT:* If $p(x)=x^3-3x^2-4x+12$ then $p(3)=0$.*]*

We need to do partial fractions and thus we need to factor the denominator. The hint suggests that $(s-3)$ is a factor in $s^3-3s^2-4s+12$. In fact, long (alternatively, synthetic) division shows $s^3-3s^2-4s+12 = (s-3)(s^2-4)$. Thus we seek A,B,C so that

$$\frac{1}{s^3-3s^2-4s+12} = \frac{A}{s-3} + \frac{B}{s-2} + \frac{C}{s+2}.$$

Clear denominators to get

$$1 \stackrel{*}{=} A(s-2)(s+2) + B(s-3)(s+2) + C(s-3)(s-2).$$

Turning this into a system of three equations in three unknowns works fine.

In this case, with no repeated roots or unfactorable quadratics (over real numbers), the following trick also works fine. Substitute $s=3,2,-2$, successively, into $*$ to get the equations $1=A(1)(5)$, $1=B(-1)(4)$, $1=C(-5)(-4)$. That is, $A=1/5$, $B=-1/4$, $C=1/20$. Thus

$$\frac{1}{s^3-3s^2-4s+12} = \frac{1}{5}\frac{1}{s-3} - \frac{1}{4}\frac{1}{s-2} + \frac{1}{20}\frac{1}{s+2}.$$

These are the Laplace transforms of exponentials. In fact, the given function of s is $\bar{f}(s)$ where

$$f(t) = \frac{1}{5}e^{3t} - \frac{1}{4}e^{2t} + \frac{1}{20}e^{-2t}.$$

10. (a) *An infinitely-long heated rod might, in equilibrium, satisfy the equation*

$$0 = k\frac{d^2T}{dx^2} + \beta T + \gamma(x). \tag{1}$$

Here $T(x)$ is the temperature of the rod at position x and you may assume $T(x)$ is positive. The constants $k > 0$ and $\beta > 0$ are the conductivity and a self-heating rate. (The latter might be for heating which results from a chemical reaction, but the meaning of these constants is not essential to this problem. You should indeed assume they are constant, however.) The term $\gamma(x)$ is external heating which we allow to depend on x.

Assuming, as needed, that $T(x)$ and its derivatives go to zero as $|x| \to \infty$, solve equation (1) using the Fourier transform. In particular, write the solution $T(x)$ as an integral in the form

$$T(x) = \int_{-\infty}^{\infty} H(x,\omega)\,\tilde{\gamma}(\omega)\,d\omega \tag{2}$$

where $\tilde{\gamma}(\omega)$ is the Fourier transform of $\gamma(x)$. That is, find a formula for $H = H(x,\omega)$.

All that is required here, in fact, is to Fourier transform ODE (1) and inspect the result. Let $\tilde{T}(\omega)$ be the Fourier transform of $T(x)$. Then

$$0 = -k\omega^2\tilde{T}(\omega) + \beta\tilde{T}(\omega) + \tilde{\gamma}(\omega).$$

Solving for $\tilde{T}$, we get

$$\tilde{T}(\omega) = \frac{1}{k\omega^2 - \beta}\,\tilde{\gamma}(\omega).$$

To recover $T(x)$ we must apply the inverse Fourier transform, and this leads to the desired form:

$$T(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{k\omega^2 - \beta}\,\tilde{\gamma}(\omega)\,e^{i\omega x}\,d\omega = \int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\frac{e^{i\omega x}}{k\omega^2 - \beta}\right]\tilde{\gamma}(\omega)\,d\omega.$$

That is,

$$H(x,\omega) = \frac{1}{\sqrt{2\pi}}\frac{e^{i\omega x}}{k\omega^2 - \beta}.$$

This is a Green's function!

(b) *By part **(a)** above, $T(x)$ is perhaps not well-defined for all functions $\gamma(x)$. In fact, describe a nonzero external heating function $\gamma(x)$ for which formula* (2) *is certainly well-defined by sketching an allowed graph of $\tilde{\gamma}(\omega)$, the Fourier transform of $\gamma(x)$, on the axes below. Specifically mention the features of $\tilde{\gamma}(\omega)$ which allow $T(x)$ to be well-defined. In particular, relate these features to the function $H(x,\omega)$ which you found in **(a)**.*

The formula from part **(a)** has an outstanding difficulty in that there is a division by zero in the integrand (i.e. in $H(x,\omega)$). In fact, as hinted in the given axes, the denominator in H is zero if

$$\omega = \pm\sqrt{\beta/k}.$$

So, a given (Fourier-transformed) heat source $\tilde{\gamma}(\omega)$ is only going to give an obviously convergent integral if it is zero in the vicinity of these singular points. Furthermore, $\tilde{\gamma}(\omega)$ must decay reasonably fast as $\gamma \to \pm\infty$.

You are only asked to graph *one* such nice function $\tilde{\gamma}(\omega)$. Here is mine: