Selected Solutions to Assignment #8

I graded E, 16.1, and 16.4 at four points each for a total of 12 points.

E. Substitute $y(x) = \sum_{n=0}^{\infty} a_n x^n$ into the given ODE to get

$$\sum_{n=0}^{\infty} \left[(n+2)(n-1)a_{n+2} + (n+1)a_{n+1} - 6a_n\right] x^n = 0$$

after a bit of re-indexing. Thus the coefficients satisfy

$$a_{n+2} = -\frac{1}{n+2} a_{n+1} + \frac{6}{(n+2)(n+1)} a_n, \qquad n \ge 0.$$

Now, we know that the general solution of the ODE is

$$y(x) = c_1 e^{-3x} + c_2 e^{2x}.$$

How do we recognize, in the mess we generate from the series method, these two exponential solutions? Note that $a_0 = y(0)$ and $a_1 = y'(0)$, while, on the other hand, if $y_1(x) = e^{-3x}$ then $y_1(0) = 1$ and $y_1'(0) = -3$.

Thus we generate the coefficients corresponding to $a_0 = 1$ and $a_1 = -3$:

$$a_2 = -(1/2)a_1 + 3a_0 = 9/2 = +3^2/2,$$
$$a_3 = -(1/3)a_2 + a_1 = -9/2 = -3^3/6 = -3^3/3!,$$
$$a_4 = -(1/4)a_3 + (1/2)a_2 = 27/8 = +3^4/4!,$$

and so on. That is, $y(x) = 1 - 3x + (3x)^2/2 - (3x)^3/3! + (3x)^4/4! - \cdots = e^{-3x}$ if $a_0 = 1$, $a_1 = -3$.

A similar calculation generates the series for $y_2(x) = e^{2x}$.

F. Into Airy's equation $\ddot{y} + ty = 0$ we substitute the usual series

$$y(t) = \sum_{n=0}^{\infty} a_n t^n. \tag{1}$$

After completely standard manipulations we get

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_{n-1}\right] t^n = 0.$$

The coefficients on the left must be zero. That is, we get a series of (simplified) equations

$$a_2 = 0 \qquad a_6 = \tfrac{-1}{6\cdot 5} \cdot \tfrac{-1}{3\cdot 2} a_0$$
$$a_3 = \tfrac{-1}{3\cdot 2} a_0 \qquad a_7 = \tfrac{-1}{7\cdot 6} \cdot \tfrac{-1}{4\cdot 3} a_1$$
$$a_4 = \tfrac{-1}{4\cdot 3} a_1 \qquad a_8 = 0$$
$$a_5 = 0 \qquad \vdots$$

with the general recursion

$$a_{n+2} = \frac{-a_{n-1}}{(n+2)(n-1)}. \tag{2}$$

We are not asked for the general solution, but rather for the solution with $y(0) = 1$ and $\dot{y}(0) = 0$. Note that from equation (1) we have $y(0) = a_0$ and $\dot{y}(0) = a_1$. Thus we have only $a_0, a_3, a_6, \dots$ as nonzero coefficients and the series is

$$y(t) = 1 - \frac{t^3}{3 \cdot 2} + \frac{t^6}{6 \cdot 5 \cdot 3 \cdot 2} - \frac{t^9}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} + \dots \tag{3}$$

There is an obvious pattern to the coefficients but no compact way to write it down that I know.

This is an Airy function. It turns out, however, that other choices of initial conditions produce the standardized Airy functions "Ai(t)" and "Bi(t);" see Abramowitz and Stegun.

Now we want to plot the partial sums. The "$N = 3, 6, 10, \dots$" instruction in the problem statement is ambiguous; here I interpret N as being the number of *nonzero* terms in (3), so, for instance, the $N = 3$ case is actually a 6th degree polynomial. Plots of the partial sums appear in figure 1.

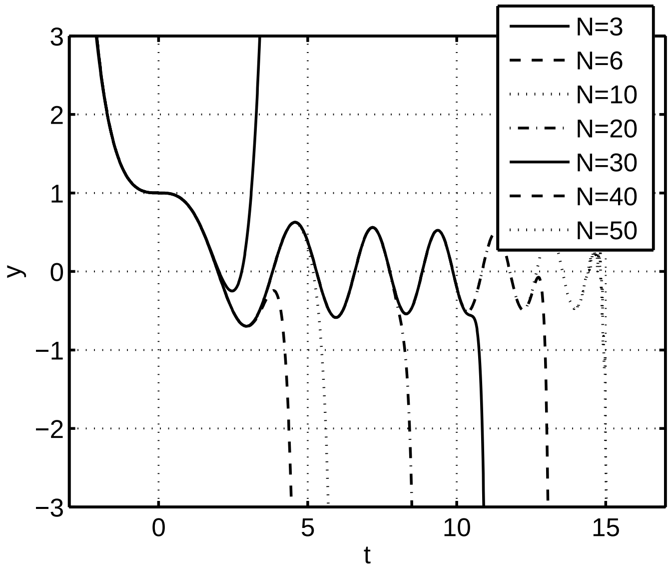

FIGURE 1. Polynomial approximations to (3); partial sums.

The MATLAB which produced this figure is online as `plotairy.m`.

We see how power series converg in figure 1. The partial sums are accurate only on an interval around the basepoint $z_0 = 0$. Note that series (3) has very rapidly decaying coefficients and thus, as one can show with the ratio test, it has interval of convergence $(-\infty, \infty)$. Also, as $t \to -\infty$ the Airy function grows very rapidly, because there is no cancellation in the series, while for $t \to \infty$ the Airy function acts like a slowly decaying sinusoid of increasing frequency. In fact, an *asymptotic* approximation is known which is accurate for $t >> 1$:

MATLAB has Airy functions built-in. They are normalized differently from our function $y(t)$ above, including with a reversal of the time axis, but one can recover the graph we want by the linear combination of the built-in functions $\mathrm{Ai}(t), \mathrm{Bi}(t)$ which satisfies the initial conditions $y(0) = 1$ and $\dot{y}(0) = 0$; see `help airy`. The code below produces figure 2, which clearly shows the function which is the full series (3).

```
>> c=[airy(0,0) airy(2,0); airy(1,0) airy(3,0)]\[1; 0];
>> t=-3:.01:17; plot(t,real(c(1)*airy(0,-t)+c(2)*airy(2,-t)))
>> xlabel t, ylabel y, axis([-3 17 -3 3]), grid on
```

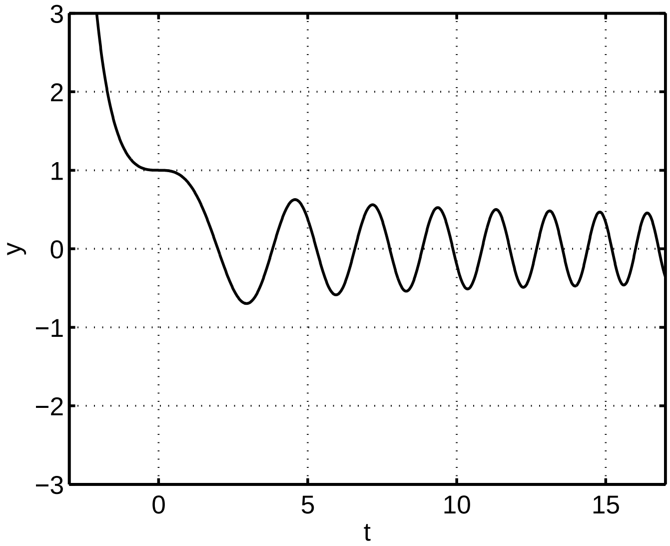

FIGURE 2. The Airy function $y(t)$ in (3), as computed by MATLAB's `airy`.

The remainder of the question asked about the actual evaluation of a function like MATLAB's `airy`. Note that MATLAB has no trouble claiming that $y(10^6) = -0.017483$ for the function defined by (3). While I don't know the accuracy of this particular result, I am confident one cannot practically get it by power series methods as above because one would have to use an Nth partial sum for a ridiculously large N. Instead, for large arguments, as described in *Numerical Recipes* in particular, the above-mentioned asymptotic approximation is used. Furthermore, instead of using a truncated power series, i.e. a polynomial approximation, it is somewhat more efficient to use a rational approximation for modest-sized inputs.

Exercise 16.1. [*I have no idea why the hint for this problem involves values of σ!*]

The basepoint z_0 is ordinary for this equation so we substitute $y(z) = \sum_{n=0}^{\infty} a_n z^n$ into the ODE and get, after some manipulation with indices,

$$[2a_0 + \lambda a_2] + [6a_3 + (\lambda - 3)a_1]z + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + (\lambda - n(n+2))\, a_n]\, z^n = 0.$$

The equations follow: once a_0, a_1 are chosen we can compute

$$a_2 = -(\lambda/2)a_0,$$
$$a_{n+2} = \frac{n(n+2) - \lambda}{(n+2)(n+1)} a_n, \quad n \geq 1.$$

In particular, we can write the general solution as a linear combination of two power series solutions

$$y(z) = a_0 \left(1 - \frac{\lambda}{2} z^2 - \frac{\lambda(8-\lambda)}{4!} z^4 - \frac{\lambda(8-\lambda)(24-\lambda)}{6!} z^6 - \dots\right)$$
$$+ a_1 \left(z + \frac{3-\lambda}{3!} z^3 + \frac{(3-\lambda)(15-\lambda)}{5!} z^5 + \dots\right)$$

The point to observe now is that one of the two series will terminate if λ is chosen to be one of the obvious constants in the numerators of the coefficients. In particular, the values

$\lambda = 0, 3, 8, 15, 24, \ldots, N(N+2), \ldots$ cause such termination. Let $N = 2$, so $\lambda = 2(2+2) = 8$, and use the first series ($a_0 = 1$, $a_1 = 0$) to define the polynomial solution

$$U_2(z) = 1 - 4z^2.$$

Let $N = 3$, so $\lambda = 3(3+2) = 15$, and use the second series ($a_0 = 0$, $a_1 = 1$) to get the polynomial solution

$$U_3(z) = z - 2z^3.$$

These are relatives of the Chebyshev polynomials, but not quite for the Chebyshev ODE. [*Despite the answers given in the text, it makes no sense to include unknowns "a_0" and "a_1" in the definitions of these polynomials.*]

Exercise 16.4. A reasonably careful change of variable would be to write $x = z - \alpha$ and define

$$g(x) := f(x + \alpha) = f(z).$$

Then $g'(x) = f'(z)$ and $g''(x) = f''(z)$, so the original ODE is equivalent to

$$g'' + 2xg' + 4g = 0.$$

We solve the ODE for $g(x)$ at the ordinary point $x_0 = 0$ by substituting $g(x) = \sum_{n=0}^{\infty} a_n x^n$ and getting, after a very little bit of processing,

$$(2a_2 + 4a_0) + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + (2n+4)a_n\right] x^n = 0.$$

Noting $2(n+2) = 2n + 4$, the coefficients satisfy

$$a_{n+2} = -\frac{2}{n+1}\, a_n, \quad n \geq 0.$$

Writing the solution as a linear combination of power series we see

$$g(x) = a_0 \left(1 - 2x^2 + \frac{2^2}{3}x^4 - \frac{2^3}{5 \cdot 3}x^6 + \frac{2^4}{7 \cdot 5 \cdot 3}x^8 - \ldots\right) + a_1 \left(x - \frac{2}{2}x^3 + \frac{2^2}{4 \cdot 2}x^5 - \frac{2^3}{6 \cdot 4 \cdot 2}x^7 + \ldots\right).$$

It is the latter series which is recognizable, because we can cancel powers of 2 and we can factor an x:

$$x - \frac{2}{2}x^3 + \frac{2^2}{4 \cdot 2}x^5 - \frac{2^3}{6 \cdot 4 \cdot 2}x^7 + \cdots = x\left(1 + (-x^2) + \frac{1}{2 \cdot 1}(-x^2)^2 + \frac{1}{3 \cdot 2 \cdot 1}(-x^2)^3 + \ldots\right) = xe^{-x^2}$$

The other series is not recognizable to me.

Once one replaces $x = z - \alpha$ then the solution can be written in the summation form given in the text, but my solution would be presented

$$f(z) = a_1\,(z - \alpha)e^{-(z-\alpha)^2} + a_0 \sum_{k=0}^{\infty} \frac{(-2)^k}{(2k-1) \cdot (2k-3) \ldots 3 \cdot 1}(z - \alpha)^{2k}.$$