Selected Solutions to Assignment #7

I graded 15.16, 15.23, 15.24b, 15.27, and 15.31 at 4 points each for a total of 20 points.

Exercise 15.16. The equation given simplifies to

$$(c-d)u_{n+1} + 2d\,u_n - (c+d)u_{n-1} = 0$$

where $c, d > 0$. We will need to assume $c \neq d$ in what follows. (*The actual situation when numerically-solving a differential equation in a reliable way would be $c \ll d$. We are asked, also, to show what happens when $c > d$.*) The characteristic equation for this recurrence relation, corresponding to $u_n = \lambda^n$, is $(c-d)\lambda^2 + 2d\lambda - (c+d) = 0$. This equation has roots

$$\lambda = \frac{-2d \pm \sqrt{4d^2 + 4(c-d)(c+d)}}{2(c-d)} = \left\{1, \frac{d+c}{d-c}\right\}.$$

(*It is not surprising that one root is 1, in retrospect, because the sum of all coefficients in the original recursion is zero.*)

The general solution to the recursion is

$$u_n = c_1 + c_2((d+c)/(d-c))^n.$$

The boundary conditions show $c_1 + c_2 = 0$ and $c_1 + c_2((d+c)/(d-c))^M = 1$, and we get after a calculation

$$u_n = \frac{-(d-c)^M + (d-c)^M\,((d+c)/(d-c))^n}{(d+c)^M - (d-c)^M} = \frac{-(d-c)^M(d-c)^n + (d-c)^M(d+c)^n}{(d-c)^n\,[(d+c)^M - (d-c)^M]};$$

it is easy to check that this satisfies the boundary conditions and the recursion.

We want to consider the ratio of successive values, and show that if $c > d$ then this ratio is negative (so successive values have opposite signs). But

$$\frac{u_{n+1}}{u_n} = \frac{-(d-c)^M + (d-c)^M\,((d+c)/(d-c))^{n+1}}{-(d-c)^M + (d-c)^M\,((d+c)/(d-c))^n} = \frac{((d+c)/(d-c))^{n+1} - 1}{((d+c)/(d-c))^n - 1}.$$

Now, if $c, d > 0$ and $c > d$ then $(d+c)/(d-c) < -1$. If n is even, for example, it then follows that the denominator of the above fraction is positive while the numerator is negative. The opposite holds if n is odd. Thus the ratio is odd.

A CONTEXT FOR THIS PROBLEM: *Consider the boundary value problem for the equilibrium temperature distribution $u = u(x)$ in a material with conductivity $K > 0$. Suppose the material is also moving uniformly with constant velocity v and that the boundary temperatures are specified. The problem is then*

$$v\frac{\partial u}{\partial x} = K\frac{\partial^2 u}{\partial x^2}, \qquad u(0) = 0, \quad u(1) = 1.$$

This is called an conduction/advection *problem. Suppose one approximates by finite differences, letting $\Delta x = 1/M$ and $x_n = n\Delta x$. Suppose $u_n \approx u(x_n)$. Then $u_0 = 0$ and $u_M = 1$ by the boundary conditions. We replace*

$$\frac{\partial u}{\partial x} \to \frac{u_{n+1} - u_{n-1}}{2\Delta x}, \qquad \frac{\partial^2 u}{\partial x^2} \to \frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta x^2};$$

these are the standard centered-difference approximations. The differential equation is replaced by

$$v\frac{u_{n+1} - u_{n-1}}{2\Delta x} = K\frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta x^2}.$$

With the identifications $c = v/(2\Delta x)$ and $d = K/(\Delta x^2)$, this is exactly the recursion equation which is the subject of the exercise. Note Δx is presumably small, and, in particular, as $\Delta x \to 0$ it is eventually the case that $c \ll d$ as long as $v > 0$ and $K > 0$.

The solution of the differential equation can actually be found by hand. (Thus this is a textbook example and not a full "real-world" problem.) On the other hand, how accurate is the finite difference solution u_n? Clearly the answer depends on M, and, in theory, the exact solution is recovered in the limit $M \to \infty$. But, in fact, the last comment in the original exercise gives a minimum *M for which the approximate solution u_n is even qualitatively right. That is, if $c > d$ then there is a qualitatively wrong fact about u_n: it oscillates.*

In the context of numerical solution of differential equations, we have to interpret this oscillation as instability. *See figure 1. Note that when $M = 6$ we have $c > d$, and the solution u_n is clearly wrong, while for $M = 16$ we have $c < d$ and there is a reasonable solution (though still not the* exact *solution).*

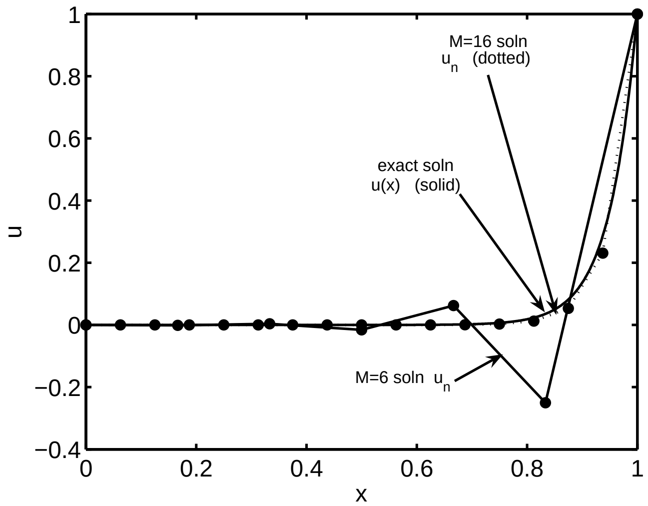

FIGURE 1. Stable and unstable finite difference solutions to an convection/advection boundary value problem.

By the way, the MATLAB *to produce this figure is as follows:*

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v=10; K=1/2; x=0:.001:1; uex=(exp(v*x/K)-1)/(exp(v/K)-1);
M=6; dx=1/M; c=v/(2*dx); d=K/dx^2; xn=0:dx:1;
lam=(d+c)/(d-c); p1=(d-c)^M; p2=(d+c)^M; n=0:M; un=(-p1+p1*lam.^n)/(p2-p1);
plot(x,uex,xn,un,'.-')
M=16; dx=1/M; c=v/(2*dx); d=K/dx^2; xn=0:dx:1;
lam=(d+c)/(d-c); p1=(d-c)^M; p2=(d+c)^M; n=0:M; un=(-p1+p1*lam.^n)/(p2-p1);
hold on, plot(xn,un,'.:'), hold off
xlabel x, ylabel u
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Exercise 15.23. The hint provided for this solution is the "straight man" answer to this question.

An equally correct, but cheeky, answer is to note that for $x \neq 2$ this is a second order linear homogeneous ODE and so we need only verify that two linearly-independent solutions have been given to show that we have the general solution.

In particular, the displayed general solution can be written

$$y(x) = k\, y_1(x) + c\, y_2(x) \qquad \text{where} \qquad y_1(x) = \frac{1}{(x-2)^2}\left(\frac{2}{3x} - \frac{1}{2}\right), \quad y_2(x) = \frac{x^2}{(x-2)^2}.$$

Substitution of y_1 into the differential equation is a completely tedious exercise, but is simply calculus. Similarly for the substitution of y_2.

What remains after such substitutions is to show that the two solutions are linearly-independent. This case, in which one has acquired the solutions by unspecified means, is a case in which the

Wronskian is a useful concept. Indeed, the Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

is nonzero if and only if y_1 and y_2 are linearly-independent. Computing the Wronskian is merely tedious.

In this case we can see that the solutions are linearly-independent more directly. Namely, y_1 and y_2 are linearly-*dependent* only if their numerators are linearly-dependent, because y_1 and y_2 have the same denominator. Thus linear-dependence would only be true if there were nonzero constants a, b so that

$$a\left(\frac{2}{3x} - \frac{1}{2}\right) + b\,x^2 = 0$$

for all $x \neq 2$. If $x = 0$ this relation shows $a = 0$, while if $x = 4/3$ this relation shows $b = 0$, so linear-dependence is impossible.

Exercise 15.24b. Here we do the whole story: First we find the general solution to the homogeneous problem:

$$y_h = c_1\,e^x + c_2\,xe^x.$$

This follows from the characteristic polynomial $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$ so $y_1 = e^x$ is one solution and, because of the repeated root, $y_2 = xe^x$ is the another linearly-independent solution.

Now we seek a particular solution of the form

$$y_p = k_1(x)y_1(x) + k_2(x)y_2(x),$$

that is, by variation of parameters, and we get a system of equations for k_1', k_2':

$$\begin{aligned} k_1'\,e^x + k_2'\,xe^x &= 0, \\ k_1'\,e^x + k_2'\,(1+x)e^x &= 2xe^x. \end{aligned}$$

[*I find it worthwhile to think through the derivation of this result!*] It is easy to solve, for instance by subtracting the equations, to get $k_2' = 2x$ so $k_2(x) = x^2$. (Note we need not keep any constants of integration at this stage because that will merely repeat a "piece of" a homogeneous solution.) Then $k_1' = -xk_2' = -2x^2$ so $k_1(x) = -(2/3)x^3$. Thus

$$y_p(x) = -\frac{2}{3}x^3e^x + x^2\,xe^x = \frac{1}{3}x^3e^x.$$

This is straightforward to check (with modest product rule skills!). The general solution is

$$y(x) = c_1\,e^x + c_2\,xe^x + \frac{1}{3}x^3e^x.$$

Needless to say, this result can also be achieved by the method of undetermined coefficients.

Exercise 15.27. This problem asks us to do the Green's function calculation again, but this time in the abstract. It turns out *not* to be harder than previous concrete examples.

We are told to assume that we have linearly-independent solutions y_1 and y_2. At the appropriate time we will use the facts $y_1(0) = 0$ and $y_2(1) = 0$. For now, let's write down $G(x,\xi)$ for the problem consisting of the given ODE and the boundary values $y(0) = 0$ and $y(1) = 0$:

$$G(x,\xi) = \begin{cases} c_1y_1(x) + c_2y_2(x), & 0 < x < \xi, \\ d_1y_1(x) + d_2y_2(x), & \xi < x < 1. \end{cases}$$

We have introduced four unknown constants and thus need four conditions. First, the boundary conditions *and the given facts about* y_1, y_2 imply

$$0 = c_1 \cdot 0 + c_2 y_2(0) \qquad \text{and} \qquad 0 = d_1 y_1(1) + c_2 \cdot 0.$$

Assuming, generically, that $y_2(0)$ and $y_1(1)$ *are not zero*—in fact we can prove they are not; how?—we get $c_2 = 0$ and $d_1 = 0$ so

$$G(x,\xi) = \begin{cases} c_1 y_1(x), & 0 < x < \xi, \\ d_2 y_2(x), & \xi < x < 1. \end{cases}$$

Next, continuity of $G(x,\xi)$ at $x = \xi$ implies $c_1 y_1(\xi) = d_2 y_2(\xi)$. The usual jump condition on the first derivative of G—it's worth rederiving!—says

$$\frac{\partial G}{\partial x}(\xi^+,\xi) - \frac{\partial G}{\partial x}(\xi^-,\xi) = d_2 y_2'(\xi) - c_1 y_1'(\xi) = 1.$$

Rewriting the last two equations, we have a system of two equations in two unknowns for c_1, d_2:

$$\begin{aligned} y_1(\xi)c_1 - y_2(\xi)d_2 &= 0, \\ -y_1'(\xi)c_1 + y_2'(\xi)d_2 &= 1. \end{aligned}$$

Combining these equations leads us to the Wronskian whether we like it or not:

$$d_2 = y_1(\xi)/W(\xi), \qquad c_1 = y_2(\xi)/W(\xi),$$

where $W(\xi) = y_1(\xi)y_2'(\xi) - y_1'(\xi)y_2(\xi)$. This is the desired result.

Exercise 15.31. The solution to the homogeneous equation $\ddot{x} + \alpha\dot{x} = 0$ is $x_h(t) = c_1 + c_2 e^{-\alpha t}$. Thus the Greens's function which solves $\ddot{x} + \alpha\dot{x} = \delta(t - t_0)$, for $t_0 > 0$ and $t > 0$, is

$$G(t,t_0) = \begin{cases} c_1 + c_2 e^{-\alpha t}, & 0 \le t < t_0, \\ d_1 + d_2 e^{-\alpha t}, & t_0 < t < \infty. \end{cases}$$

The initial conditions only apply to the first case of the formula for the Green's function: $c_1 + c_2 = 0$ and $-\alpha c_2 = 0$ imply $c_1 = c_2 = 0$. The continuity of $G(t, t_0$ at $t = t_0$ implies $0 = d_1 + d_2 e^{-\alpha t}$. The jump condition

$$\frac{\partial G}{\partial t}(t_0+, t_0) - \frac{\partial G}{\partial t}(t_0^-, t_0) = 1$$

becomes $-\alpha d_2 e^{-\alpha t_0} = 1$. We conclude

$$G(t,t_0) = \begin{cases} 0, & 0 \le t < t_0, \\ \alpha^{-1}\left(1 - e^{\alpha(t_0 - t)}\right), & t_0 < t < \infty, \end{cases}$$

and that

$$x(t) = \int_0^\infty G(t,t_0)f(t_0)\,dt_0 = \int_0^t \alpha^{-1}\left(1 - e^{\alpha(t_0-t)}\right) f(t_0)\,dt_0$$

is the solution to the general nonhomogeneous equation $\ddot{x} + \alpha\dot{x} = f(t)$.

Now, when $f(t) = Ae^{-\alpha t}$ we can do the integral:

$$\begin{aligned} x(t) &= \int_0^t \alpha^{-1}\left(1 - e^{\alpha(t_0-t)}\right) Ae^{-at_0}\,dt_0 = A\alpha^{-1}\int_0^t e^{-at_0} - e^{-\alpha t}e^{t_0(\alpha - a)}\,dt_0 \\ &= A\alpha^{-1}\left[a^{-1}(1 - e^{-at}) - e^{-\alpha t}(a-\alpha)^{-1}(1 - e^{t(\alpha - a)})\right] \\ &= A(\alpha - a)^{-1}\left[a^{-1}(1 - e^{-at}) - \alpha^{-1}(1 - e^{-\alpha t})\right]. \end{aligned}$$

The last two forms are not *obviously* equivalent, but after some work I saw that the form I computed (the first) matches the solution in the text.