Selected Solutions to Assignment #6

I graded none of these!

Exercise B. The given column vector represents, as a Taylor series,

$$f(x) = 1 + 2!x + 3!x^2 + \cdots + (n+1)!x^n + \dots$$

This series converges (trivially) for $x = 0$, but it turns out that it does not converge for any other value of x. In fact, applying the ratio test (subsection 4.3.2 of Riley, Hobson, & Bence),

$$\rho = \lim_{n\to\infty} \frac{(n+2)!|x|^{n+1}}{(n+1)!|x|^n} = \lim_{n\to\infty} (n+2)|x| = +\infty.$$

(Unless $x = 0$, as noted.) Thus the series has radius of convergence zero. It represents no meaningful function of x.

The same ratio test applied to

$$g(x) = 4 + 5 \cdot 3x + 6 \cdot 3^2x^2 + \cdots + (n+4) \cdot 3^n x^n + \dots$$

gives a radius of convergence 1/3. Indeed,

$$\rho = \lim_{n\to\infty} \frac{(n+5)3^{n+1}|x|^{n+1}}{(n+4)3^n|x|^n} = 3|x| \lim_{n\to\infty} \frac{n+5}{n+4} = 3|x|,$$

Thus $\rho < 1$, which is required for the ratio test to imply convergence, is equivalent to $|x| < 1/3$ or $-1/3 < x < 1/3$. The given coefficients for g imply that $g(x)$ is a meaningful function on the interval $(-1/3, 1/3)$ but we would not get anything useful evaluating $g(x)$ at $x = 2$.

A generic problem with the "Taylor coefficients as a column vector" representation scheme is that many such column vectors do not represent functions on the whole real line, nor, necessarily, any interval at all. In particular, *adding* two such "vectors" gives a function which converges on the smaller interval, so elaborate computations might well yield arbitrarily small intervals on which the result is defined. Another problem is that it is not clear how the magnitude of one of these "column vectors" could/should be defined; there is not much chance of a Parseval-like result relating a norm computed from the function with a norm computed with the coefficients in the column vector.

Exercise C. We do the computation for $\tilde{L}$ which we did for L, that is, we apply the operator to $\sin nx$ and $\cos nx$:

$$\tilde{L}(\sin nx) = \frac{d^2}{dx^2}(\sin nx) + \frac{d}{dx}(\sin nx) + 4(\sin nx) = (-n^2+4)\sin nx + n\cos nx,$$
$$\tilde{L}(\cos nx) = \frac{d^2}{dx^2}(\cos nx) + \frac{d}{dx}(\cos nx) + 4(\cos nx) = (-n^2+4)\cos nx - n\sin nx.$$

Thus, using the ordered basis in the second representation scheme,

$$\tilde{L} = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 3 & -1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 3 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & -5 & -3 & \\ 0 & 0 & 0 & 0 & 0 & 3 & -5 & \\ & & \vdots & & & & & \ddots \end{pmatrix}$$

It is not *diagonal*; I am sorry that the hint is wrong![1]

Actually, row operations can give a diagonal matrix with nonzero diagonal. In fact, if R_n denotes the nth row then the following *sequence* of operations produce a diagonal matrix: $R_3 \longleftarrow (-1/3)R_2 + R_3$; $R_2 \longleftarrow R_2 + (3/10)R_3$; $R_5 \longleftarrow R_4$; $R_4 \longleftarrow R_5$; $R_7 \longleftarrow (3/5)R_6 + R_7$; $R_6 \longleftarrow (-9/24)R_7 + R_6$; ... The result is a diagonal matrix with nonzero diagonal:

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 10/3 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & -5 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & -24/9 & \\ & & \vdots & & & & & \ddots \end{pmatrix}$$

If working with infinity-by-infinity matrices were generally reliable (it's not!) then the above row operations would show that $\tilde{L}$ is invertible. But that would be bad! We would then conclude that the ODE $y'' + y' + 4y = f(x)$ had a unique solution *without* any initial values, and this is not true.

The reason that L has nullity 2 while $\tilde{L}$ has nullity zero (if one trusts the above arguments) *has everything to do with boundary values.* In particular, the solutions of $Ly = 0$ are periodic with period 2π, which is also true of the basis we are using. Thus, in particular, the matrix representation of L in this basis has zeros on the diagonal because the solutions of $Ly = 0$ are included among the finite linear combinations of the basis elements. By contrast, the solutions of $\tilde{L}y = 0$, namely $y_1(x) = e^{-x/2}\cos(\sqrt{15}x/2)$ and $y_2(x) = e^{-x/2}\sin(\sqrt{15}x/2)$, are not periodic with period 2π. Because we have no smooth representation of these functions (in particular), the entries of the matrix for $\tilde{L}$ do not directly reflect their span. That is, the kernel of $\tilde{L}$ is not at all apparent in the matrix entries.

Exercise D. Here we compute

$$L(\sin n\pi x) = \left[-(n\pi)^2 + 4\right]\sin n\pi x,$$

so

$$L = \begin{pmatrix} -\pi^2+4 & 0 & 0 & 0 & \\ 0 & -4\pi^2+4 & 0 & 0 & \\ 0 & 0 & -9\pi^2+4 & 0 & \cdots \\ 0 & 0 & 0 & -16\pi^2+4 & \\ & & \vdots & & \ddots \end{pmatrix}$$

On the other hand, the Fourier sine series holds (for $|x| < 1$):

$$x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\pi} \sin n\pi x.$$

[1] I was just plain wrong. The next paragraph sort of recovers the idea, but I did not intend to need it. I should have defined "$\tilde{L} = d^2/dx^2 + 5$," for instance.

Thus we want to solve the matrix problem

$$\begin{pmatrix} -\pi^2+4 & 0 & 0 & 0 & \\ 0 & -4\pi^2+4 & 0 & 0 & \\ 0 & 0 & -9\pi^2+4 & 0 & \cdots \\ 0 & 0 & 0 & -16\pi^2+4 & \\ & & \vdots & & \ddots \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 2/\pi \\ -2/(2\pi) \\ 2/(3\pi) \\ -2/(4\pi) \\ \vdots \end{bmatrix}$$

The solution is easy because the matrix is diagonal:

$$a_n = \frac{(-1)^{n+1}2/(n\pi)}{-n^2\pi^2+4}.$$

Note one never divides by zero. The solution to $Ly = x$ is

$$y(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2/(n\pi)}{-n^2\pi^2+4} \sin n\pi x.$$

This really does solve $y'' + 4y = x$, $y(0) = 0$, and $y(1) = 0$ as one can check. All we have really done here is to *see* a Fourier series calculation as a matrix calculation. This is good: it allows us to generalize!

Note that if "4" in the above linear operator were replaced with $k^2\pi^2$ for some integer k then we would end up dividing by zero. Such an apparently technical difficulty in the calculation reflects a real problem. In fact, we will see that the boundary value problem

$$y'' + \pi^2 y = x, \qquad y(0) = 0, \qquad y(1) = 0,$$

for example, *has no solutions.*

Exercise 14.11. Dividing the given equation by x we have

$$\left(\frac{y}{x} - 1\right)\frac{dy}{dx} + 2 + 3\frac{y}{x} = 0,$$

which is homogeneous. We let $v = y/x$ or, equivalently $y = vx$. In fact, the latter statement implies $dy/dx = x\,dv/dx + v$ by the product rule. The ODE becomes

$$(v-1)\left(x\frac{dv}{dx} + v\right) + 2 + 3v = 0;$$

we expect this to be separable. Indeed,

$$x\frac{dv}{dx} = -\frac{2+3v}{v-1} - v = \frac{2+2v+v^2}{1-v}.$$

Separated and integrated, this is

$$\int \frac{1-v}{2+2v+v^2}\,dv = \int \frac{dx}{x} = \ln|x| + C_0.$$

The left hand antiderivative requires some manipulation:

$$\begin{aligned} \int \frac{1-v}{2+2v+v^2}\,dv = \int \frac{-(v+1)+2}{(v+1)^2+1}\,dv = \int \frac{-u+2}{u^2+1}\,du &= -\frac{1}{2}\int \frac{2u}{u^2+1}\,du + 2\int \frac{1}{u^2+1}\,du \\ &= -\frac{1}{2}\int \frac{dw}{w} + 2\arctan u = -\frac{1}{2}\ln|w| + 2\arctan u \\ &= -\frac{1}{2}\ln((v+1)^2+1) + 2\arctan(v+1), \end{aligned}$$

under the substitutions $u = v + 1$ and $w = u^2 + 1$.

The simplified (implicitly-defined) solution in terms of v is now

$$4\arctan(v+1) = 2\ln|x| + \ln((v^2+1)+1) + C_1 = \ln(x^2) + \ln((v^2+1)+1) + C_1.$$

Exponentiated and with the replacement $v = y/x$, we have the implicitly-defined solution

$$\exp(4\arctan((y/x)+1)) = A\left(x^2 + (x+y)^2\right).$$

[*A final note. The "solution" above is only as useful as one's ability to use it to find y given x or vice versa. In fact, it is frequently not very useful to have such a "solution". A numerical or other approximate solution to the original ODE is likely to be just as desirable.*]

Exercise 14.13. [*This is yet another problem for which I don't quite understand the motivation. I'm glad I'm not grading it!*]

Consider

$$t\dot{y} + (t-1)y \overset{*}{=} 0;$$

we seek $y = y(t)$. The Laplace transform of $*$ is

$$-\frac{d}{ds}\left[s\bar{y}(s) - y(0)\right] - \frac{d}{ds}\bar{y}(s) - \bar{y}(s) = 0$$

or

$$-\bar{y}(s) - s\bar{y}'(s) - \bar{y}'(s) - \bar{y}(s) = 0$$

or

$$(s+1)\bar{y}'(s) + 2\bar{y}(s) = 0$$

or

$$\bar{y}'(s) + \frac{2}{s+1}\bar{y}(s) = 0.$$

In the above equations the prime is d/ds, of course. I have used formulas (13.62) and (13.57).

The last equation is linear and has integrating factor

$$\mu(s) = e^{\int 2\,ds/(s+1)} = e^{2\ln(s+1)} = (s+1)^2,$$

at least for $s > -1$ (which is certainly appropriate given that this is the Laplace transform variable). Then, as promised,

$$\bar{y}(s) = \mu^{-1}(s)C = C(s+1)^{-2}.$$

By table 13.1,

$$y(t) = Cte^{-t}.$$

On the one hand, now, $y'(0) = C$. On the other hand, if $y(t) = t + \sum_{n=2}^{\infty} a_n t^n$ then we know $y'(0) = 1$. But then $C = 1$. Also, the well-known Taylor series

$$e^{-t} = 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k$$

implies

$$y(t) = te^{-t} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{k+1} = t + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} t^n.$$

That is, $a_n = (-1)^{n-1}/(n-1)!$; *the text is wrong.*

On the other hand, we can solve $*$ directly but writing it in standard linear form

$$\dot{y} + (1 - t^{-1})y = 0,$$

noting

$$\mu(t) = e^{\int 1 - t^{-1}\,dt} = e^{t - \ln t} = t^{-1}e^t,$$

and getting

$$y(t) = \mu^{-1}(t)C = Cte^{-t}.$$

To determine C by this route, note that if we know $y(t) = t + \sum_{n=2}^{\infty} a_n t^n$ then we know $y'(0) = 1$. But then

$$1 = y'(0) = C(e^{-0} - 0e^{-0}) = C,$$

so $y(t) = te^{-t}$.

[*What was the point of all that?!*]

Exercise 15.1. Before *thinking* about the meaning of any of this, we are asked to solve the nonhomogeneous linear initial value problem

$$\ddot{x} + \omega_0^2 x = \cos \omega t, \qquad x(0) = 0, \qquad \dot{x}(0) = 0.$$

The homogeneous solution is

$$x_h(t) = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t.$$

We seek a particular solution by the "method of undetermined coefficients". We substitute is $x_p(t) = A \sin \omega t + B \cos \omega t$ and solve

$$A(\omega_0^2 - \omega^2) \sin \omega t + B(\omega_0^2 - \omega^2) \cos \omega t = \cos \omega t$$

for A and B. We see that $A = 0$ and $B = (\omega_0^2 - \omega^2)^{-1}$. Thus

$$x(t) = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t + \frac{1}{\omega_0^2 - \omega^2} \cos \omega t.$$

The initial values determine c_1, c_2: $0 = x(0) = c_2 + (\omega_0^2 - \omega^2)^{-1}$, $0 = \dot{x}(0) = \omega_0 c_1$, so

$$x(t) = \frac{\cos \omega t - \cos \omega_0 t}{\omega_0^2 - \omega^2}.$$

Now, this solution deserves a little analysis. For small t we can use Taylor series

$$\cos \theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \dots$$

to see that

$$x(t) = \frac{1 - \frac{\omega^2 t^2}{2} + \frac{\omega^4 t^4}{4!} - \dots - \left[1 - \frac{\omega_0^2 t^2}{2} + \frac{\omega_0^4 t^4}{4!} - \dots\right]}{\omega_0^2 - \omega^2} = \frac{1}{2} t^2 - \frac{\omega_0^2 + \omega^2}{4!} t^4 + \dots$$

Thus the solution starts by growing in a manner $(\frac{1}{2} t^2)$ independent of the relative values of ω and ω_0. Unfortunately, over the time scale of a cycle of these cosines, this Taylor expansion is not useful.

A different kind of analysis comes from a bit of trigonometric manipulation:

$$\begin{aligned} \cos \omega t - \cos \omega_0 t &= \cos \omega t - \cos(\omega t + (\omega_0 - \omega)t) = \cos \omega t - \cos \omega t \cos((\omega_0 - \omega)t) + \sin \omega t \sin((\omega_0 - \omega)t) \\ &\overset{*}{=} \cos \omega t \, [1 - \cos((\omega_0 - \omega)t)] + \sin \omega t \sin((\omega_0 - \omega)t) \end{aligned}$$

Now we can address the case where ω and ω_0 are close. That is, we can address resonance. Even if t is fairly large, $(\omega_0 - \omega)t$ can be quite small near resonance. If $\epsilon = \omega_0 - \omega$ and if ϵt is small then

$$\cos \omega t - \cos \omega_0 t \approx \frac{\epsilon^2 t^2}{2} \cos \omega t + \epsilon t \sin \omega t$$

by the Taylor expansions of cos and sin. If ϵ is small then ϵ^2 is much smaller. Thus the solution $x(t)$ is roughly

$$x(t) \sim (\omega_0 - \omega) t \sin \omega t.$$

That is, the solution is (for a significant time, at least) a growing sinusoid at the driving frequency.

One can see from the form $*$ of the solution that over the long term, for $t \gg \epsilon^{-1}$, there is periodic response which has the form of a "beat", that is, a low frequency modulation of a high frequency sinusoid.

Exercise 15.8. This problem can be done by Laplace transforms, but completely elementary methods work fine too. In particular, the pair of first order equations for $x(t)$ and $y(t)$ can easily be reduced to single second order equations for either.

For instance, if one takes the derivative of the first equation one has

$$\ddot{x} - 2\dot{y} = -\cos t.$$

The second equation can be solved for $\dot{y}$ (i.e. $\dot{y} = 5\cos t - 2x$) and substituted into the above to get $\ddot{x} - 2(5\cos t - 2x) = -\cos t$ or

$$\ddot{x} + 2^2 x \overset{*}{=} 9\cos t.$$

Let's solve equation $*$. The homogeneous solution is $x_h(t) = c_1 \sin 2t + c_2 \cos 2t$. The method of undetermined coefficients suggests the particular form $x_p(t) = A\sin t + B\cos t$, and substitution into $*$ leads to $A = 0$ and $B = 3$. Thus

$$x(t) = c_1 \sin 2t + c_2 \cos 2t + 3\cos t.$$

The value $x(0) = 3$ implies $c_2 = 0$. But how to use "$y(0) = 2$"? One answer is to note that the first of the original two equations, $\dot{x} - 2y = -\sin t$, can be solved for $y(t)$:

$$y(t) = \frac{\dot{x} + \sin t}{2} \qquad \Longrightarrow \qquad 2 = y(0) = \frac{\dot{x}(0) + \sin 0}{2} = c_1.$$

Thus

$$x(t) = 2\sin 2t + 3\cos t$$

and, using the recent expression for y,

$$y(t) = 2\cos 2t - \sin t.$$

So now we have a parametric description of the solution, which tends to be the most useful form. Figure 1 provides a "sketch."

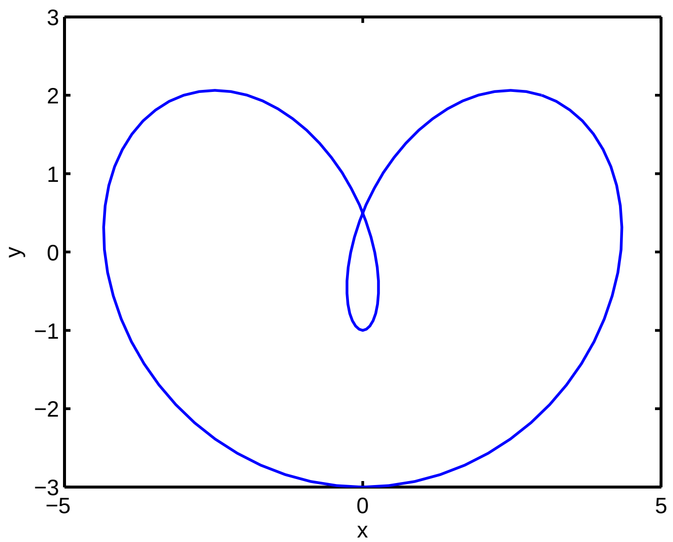

FIGURE 1. Exercise 15.8: MATLAB loves you, too.