Selected Solutions to Assignment #10

I graded G, H, I, J, and L at four points each for a total of 20 points.

G. If n is odd then $\int_0^{2\pi} \cos^n\theta\,d\theta = 0$ by the symmetry illustrated in figure 1. In fact,

$$\int_\pi^{2\pi} \cos^n\theta\,d\theta = \int_0^\pi \left(\cos(\theta'+\pi)\right)^n\,d\theta' = \int_0^\pi \left(-\cos(\theta')\right)^n\,d\theta' = -\int_0^\pi \cos^n\theta\,d\theta,$$

by substituting $\theta' = \theta+\pi$, so the combined integral over $[0,2\pi]$ is zero.

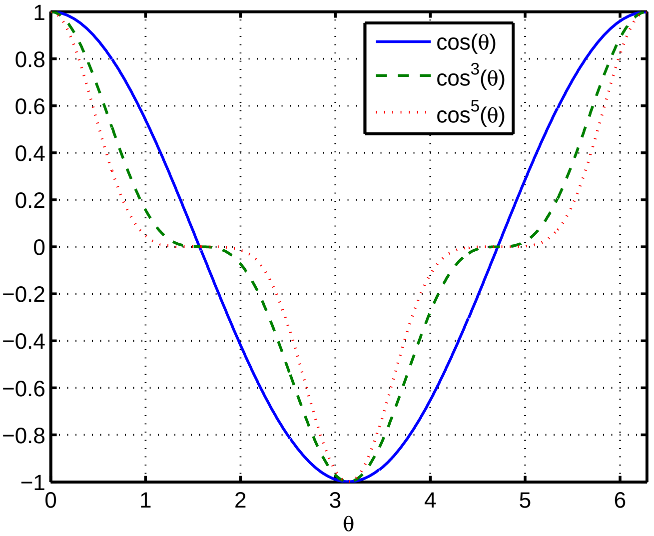

FIGURE 1. The integral of an odd power of $\cos\theta$ over $[0,2\pi]$ is zero.

If n is even then we do an integration-by-parts to get a recursion:

$$\begin{aligned}\int_0^{2\pi} \cos^{2k}\theta\,d\theta &= \int_0^{2\pi}\cos^{2k-1}\theta\,\cos\theta\,d\theta = \cos^{2k-1}\theta\,\sin\theta\Big]_0^{2\pi} - \int_0^{2\pi}(2k-1)\cos^{2k-2}\theta(-\sin\theta)\,\sin\theta\,d\theta\\ &= 0 + (2k-1)\int_0^{2\pi}\cos^{2k-2}\theta(1-\cos^2\theta)\,d\theta\\ &= (2k-1)\int_0^{2\pi}\cos^{2k-2}\theta\,d\theta - (2k-1)\int_0^{2\pi}\cos^{2k}\theta\,d\theta.\end{aligned}$$

With I_k defined in the problem statement,

$$I_k = (2k-1)I_{k-1} - (2k-1)I_k \qquad\iff\qquad I_k = \frac{2k-1}{2k}I_k.$$

Since $I_0 = 2\pi$ by a trivial integral, one finds from this recursion that

$$I_1 = \frac{2\pi}{2},\qquad I_2 = \frac{2\pi\cdot 3}{4\cdot 2} = \frac{2\pi\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 2\cdot 4\cdot 2} = 2\pi\frac{4!}{2^4\,2!\,2!},\qquad \text{etc.}$$

H.

$$F'(z) = \int_0^{2\pi}\frac{\partial}{\partial z}\left(e^{iz\cos\theta}\right)\,d\theta = \int_0^{2\pi}(i\cos\theta)e^{iz\cos\theta}\,d\theta$$

and generally

$$F^{(n)}(z) = \int_0^{2\pi}(i\cos\theta)^n e^{iz\cos\theta}\,d\theta.$$

Thus

$$F^{(n)}(0) = \int_0^{2\pi} (i\cos\theta)^n \, d\theta = i^n \int_0^{2\pi} \cos\theta^n \, d\theta.$$

Problem **G** above computes these integrals, which are only nonzero for even n.

Thus by Taylor series,

$$F(z) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} z^n = \sum_{k=0}^{\infty} \frac{i^{2k} I_k}{(2k)!} z^{2k} = 2\pi \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{2^{2k} k! k! (2k)!} z^{2k} = 2\pi \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k! k!} z^{2k}.$$

This is 2π times the series for $J_0(z)$ on page 567. Thus $F(z) = 2\pi J_0(z)$, as claimed.

I.

$$\begin{aligned}
\tilde{f}(u,v) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r) e^{-i(ux+vy)} \, dx\, dy = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} f(r) e^{-irq(\cos\phi\cos\theta + \sin\phi\sin\theta)} \, r\, dr\, d\theta \\
&= \int_0^{\infty} f(r) \left[\frac{1}{2\pi} \int_0^{2\pi} e^{-irq\cos(\theta-\phi)} \, d\theta \right] r\, dr \stackrel{*}{=} \int_0^{\infty} f(r) \left[\frac{1}{2\pi} \int_0^{2\pi} e^{-irq\cos\theta'} \, d\theta' \right] r\, dr \\
&\stackrel{**}{=} \int_0^{\infty} f(r) J_0(-rq) \, r\, dr \stackrel{\dagger}{=} \int_0^{\infty} f(r) J_0(rq) \, r\, dr.
\end{aligned}$$

At step $*$ we substitute $\theta' = \theta - \phi$ and note that the integrand $e^{-rq\cos\theta'}$ is periodic with period 2π so *any* integral over an interval of length 2π gives the same result. In step $**$ we use **H**. In step $\dagger$ we note that J_0 is an even function because the Taylor series for $J_0(z)$ has only even powers.

J. Given $f(r)$. If

$$F(q) = \int_0^{\infty} f(r) J_0(rq) r\, dr$$

and if we define the corresponding function on the the plane $g(x,y) = f(\sqrt{x^2+y^2})$ then

$$\tilde{g}(u,v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(ux+vy)} f(\sqrt{x^2+y^2}) \, dx\, dy \stackrel{*}{=} \int_0^{\infty} f(r) J_0(rq) r\, dr = F(q).$$

In step $*$ we have used **I** above, and we let $r = \sqrt{x^2+y^2}$ and $q = \sqrt{u^2+v^2}$. On the other hand,

$$g(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{g}(u,v) \, e^{+i(ux+vy)} \, du\, dv \stackrel{\dagger}{=} \int_0^{\infty} F(q) J_0(rq) q\, dq.$$

To do step $\dagger$ we technically would need to repeat the calculation in **I** but with "$+irq$" replacing "$-irq$," but it is easy to see how it would come out. Thus

$$f(r) = g(x,y) = \int_0^{\infty} F(q) J_0(rq) q\, dq.$$

In other words, the Hankel transform is self-inverse because the two-variable Fourier transform is *nearly* self-inverse in general but, in particular, it *is* self-inverse on radial functions.

K. The calculation of the Laplacian ∇^2 in polar coordinates is standard, and it is almost done in section 10.11 of the text. (It may be standard, but it is still a pain in the ...)

To compute the Hankel transform of the Laplacian we integrate-by-parts twice, and we need to assume that the boundary terms are zero:

$$\begin{aligned}\int_0^\infty \left(f''(r) + r^{-1}f'(r)\right) J_0(qr) r\,dr &= f'(r)J_0(qr)r\Big]_0^\infty - \int_0^\infty f'(r)\frac{\partial}{\partial r}\left(J_0(qr)r\right)\,dr + \int_0^\infty f'(r)J_0(qr)\,dr \\ &= 0 - \int_0^\infty f'(r)\left[J_0'(qr)qr\right]\,dr = -f(r)J_0'(qr)qr\Big]_0^\infty + \int_0^\infty f(r)\left(J_0'(qr)qr\right)'\,dr \\ &\overset{*}{=} 0 + \int_0^\infty \frac{f(r)}{r}\left(z^2J_0''(z) + zJ_0'(z)\right)\,dr \overset{\dagger}{=} \int_0^\infty \frac{f(r)}{r}\left(-z^2J_0(z)\right)\,dr \\ &= -q^2\int_0^\infty f(r)J_0(qr)r\,dr = -q^2F(q).\end{aligned}$$

We substituted $z = qr$ in step $*$ to recognize Bessel's equation

$$z^2y'' + zy' + (z^2 - 0)y = 0,$$

which is solved by $y(z) = J_0(z)$, in step $\dagger$. [*In the problem statement I failed to specify the necessary assumptions to make the boundary terms go away. In essence one cannot give them more specifically* than *to say "assume the boundary terms go away."*]

L. First, I should have mentioned in the problem statement that the given rigid plate equation includes a bouyant restoring force $-\rho g u$ where ρ is the density of the displaced fluid. The displaced fluid is the earth's mantle in the geophysical context.

Let $U(q)$ be the Hankel transform of $u(r)$ and let $\bar{\Pi}_a(q)$ be the Hankel transform of the load function $\Pi_a(r)$, which describes a disc load in polar coordinates. The given rigid plate equation has Hankel transform

$$\rho g U(q) + Dq^4U(q) = \mu\bar{\Pi}_a(q),$$

because

$$\mathcal{H}[\nabla^4 u](q) = \mathcal{H}[\nabla^2(\nabla^2 u)](q) = -q^2(-q^2U(q)) = q^4U(q),$$

where "$\mathcal{H}$" stands for the Hankel transform, so

$$U(q) = \frac{\mu\bar{\Pi}_a(q)}{\rho g + Dq^4}.$$

The transform of the load is calculated as follows:

$$\begin{aligned}\bar{\Pi}_a(q) &= \int_0^\infty \Pi_a(r)J_0(rq)r\,dr = \int_0^a J_0(rq)r\,dr \overset{*}{=} q^{-2}\int_0^{aq} J_0(z)z\,dz = q^{-2}\left(zJ_1(z)\Big]_0^{aq}\right) \\ &= q^{-2}aqJ_1(aq) = q^{-1}aJ_1(aq).\end{aligned}$$

Obviously $z = rq$ in step $*$, and then we use the antiderivative stated in the problem. It follows that, as claimed,

$$U(q) = \frac{\mu a J_1(aq)}{q(\rho g + Dq^4)}.$$

Now, $u(r)$ can be recovered by another Hankel transform (since by **J** the Hankel tranform is self-inverse):

$$u(r) = \int_0^\infty \frac{\mu a J_1(aq)}{q(\rho g + Dq^4)} J_0(rq)q\,dq = a\mu\int_0^\infty \frac{J_1(aq)\,J_0(rq)}{\rho g + Dq^4}\,dq. \qquad (1)$$

Continuation of L in a geophysical context. The last formula (1) is computable in practice, though the oscillatory integral is a bit painful numerically. In any case I have had a recent excuse to compute it, and the result is shown in figure 2 as the dashed curve.

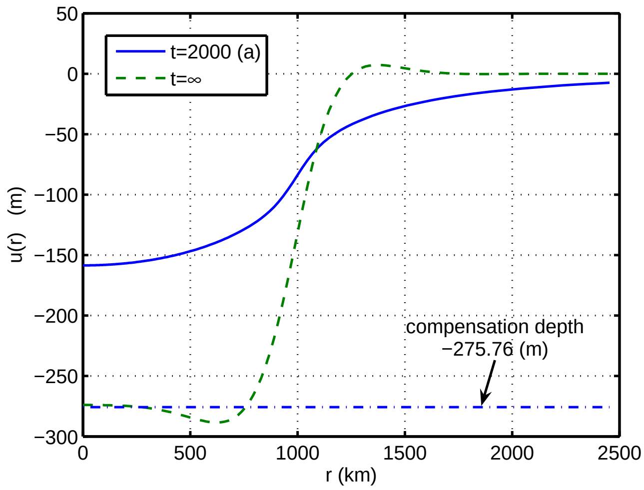

FIGURE 2. Vertical displacement at 2000 years (solid), and equilibrium displacement (at time ∞ in equation (2); dashed), for a disc load of ice with thickness 1000 (m) and radius 1000 (km). "Compensation depth" also shown; see text.

In particular, the original PDE given in **L** can be regarded as the equilibrium equation of the PDE

$$\frac{\partial}{\partial t}\left(2\eta(-\nabla^2)^{1/2}u\right) + \rho g u + D\nabla^4 u = \mu \Pi_a(r) \tag{2}$$

which describes the time-dependent displacement $u(r,t)$ of the rigid plate. Here we assume that it is underlain by a half-space filled with a viscous fluid of viscosity η. The operator $(-\nabla^2)^{1/2}$ does make sense, but the definition using the Fourier transform is skipped here; see me or David Newman if you want.

In this geophysical context[1] we have assumed that the lithosphere is a rigid plate with flexural rigidity $D = 5.0 \times 10^2 4$ N m and that the density and viscosity of the underlying mantle are $\rho = 3300\,\mathrm{kg\,m^{-3}}$ and $\eta = 10^{21}$ Pa s, respectively. The acceleration of gravity is $g = 9.81\,\mathrm{m\,s^{-2}}$.

If we suppose that a disc load of *ice*,[2] with radius $a = 1000$ km and thickness 1000 m, is placed at the origin at time $t = 0$, then figure 2 should make sense. (A separate Hankel-transform of (2) computes the result for 2000 years if one starts with zero displacement at time zero.) The equilibrium ("$t = \infty$" for equation (2)) is exactly what we solved in problem **L**. Figure 2 also shows the geophysically-interesting "bulges" which follow from (1), at roughly $r = 600$ km and $r = 1300$ km.

The "compensation depth" is the depth to which an ice disc load of thickness 1000 m and *infinite* extent would sink as $t \to \infty$; that is, this is the depth where the bouyant force balances the weight of the ice.

[1]See C. Lingle & J. Clark (1985) *A numerical model of interactions between a marine ice sheet and the solid earth: application to a West Antarctic ice stream*, J. Geophysical Research **90** (C1) 1100–1114. The constants used here are from that source.

[2]Note ice has density $\rho_i = 910\,\mathrm{kg\,m^{-3}}$. The disc load is described by $a = 1000$ km and $\mu = \rho_i g(1000\text{ m}) = 8.92 \times 10^6\,\mathrm{N\,m^{-1}}$.