Assignment #10: Hankel transforms

Due *Tuesday, 6 December 2005.*

Do the following exercises:

G. Show that

$$\int_0^{2\pi} \cos^n \theta \, d\theta = \begin{cases} 2\pi \frac{(2k)!}{2^{2k}\, k!\, k!}, & n = 2k \geq 0 \text{ even}, \\ 0, & n \geq 1 \text{ odd}. \end{cases}$$

In particular, show using integration-by-parts that if we define

$$I_k = \int_0^{2\pi} \cos^{2k} \theta \, d\theta$$

then

$$I_k = \frac{2k-1}{2k} I_{k-1}.$$

Note $I_0 = 2\pi$. (You will need to verify the odd n cases separately.)

H. Define the function

$$F(z) = \int_0^{2\pi} e^{iz\cos\theta} \, d\theta.$$

This is not an integral one can do outright. However, show that it is possible to compute $F(0)$ and, by differentiating under the integral, $F'(0)$, $F''(0)$, and so on. Show that the Taylor series for $F(z)$ at $z_0 = 0$ is

$$F(z) = \sum_{k=0}^{\infty} \frac{2\pi(-1)^k}{2^{2k}\, k!\, k!} z^{2k}.$$

(You will use the above problem.) Thereby show that

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz\cos\theta} \, d\theta,$$

as claimed in the very last equation of chapter 16. (Note that equation (16.63) defines $J_0(z)$.)

I. So what? Suppose you have a function of two variables which is actually radially-symmetric

$$f(x, y) = f(r).$$

Consider the two variable Fourier transform

$$\tilde{f}(u, v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r) e^{-i(ux+vy)} \, dx \, dy.$$

Convert this integral to polar coordinates and show that this transform of a radial function can be written

$$\tilde{f}(u, v) = \int_0^{\infty} f(r) \left[\frac{1}{2\pi} \int_0^{2\pi} e^{irq\cos(\theta)} \, d\theta \right] r \, dr = \int_0^{\infty} f(r) J_0(rq) r \, dr$$

where $q = \sqrt{u^2 + v^2}$. [*Hint*: Writing $x = r\cos\theta$, $y = r\sin\theta$, $u = q\cos\phi$, $v = q\sin\phi$ we have $ux + vy = rq\cos(\theta - \phi)$.]

J. We define the *Hankel* transform of a function $f(r)$, for $0 < r < \infty$, by

$$F(q) = \int_0^\infty f(r) J_0(rq) r\, dr.$$

Note that the Hankel transform is linear. By using the above relation to the two-variable Fourier transform, show that also

$$f(r) = \int_0^\infty F(q) J_0(rq) q\, dq.$$

This shows that the Hankel transform is self-inverse (as stated on p. 465, chapter 13).

K. (Extra Credit). Recall that if $f = f(x, y)$ then

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Show that if $f = f(r)$ is a function of r only then

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r}.$$

Continuing to suppose that $f = f(r)$, let $F(q)$ be the Hankel transform of $f(r)$. Show that the Hankel transform of $\nabla^2 f$ is $-q^2 F(q)$. (You will need to assume that $f(r)$ and its derivative are bounded functions, for instance.)

L. Let $\Pi_a(r)$ be the function which is one for $0 < r < a$ and zero for $a < r$.

A rigid plate with a *disc load* might satisfy the equation

$$\rho g u + D\nabla^4 u = \mu \Pi_a(r)$$

for $u = u(x, y)$ the deflection of the plate from the horizontal. Here ρ, g, D, μ are all positive constants. Also $\nabla^4 f = \nabla^2(\nabla^2 f)$ by definition; you may want to write out its expression in cartesian coordinates just for fun. Because the disc load $\mu\Pi_a(r)$ is radially-symmetric, and assuming the plate is infinite in extent, we may assume $u = u(r)$. Use the result of the previous problem to show that if $U(q)$ is the Hankel transform of $u(r)$ then

$$U(q) = \frac{a\mu J_1(aq)}{q\,(\rho g + D q^4)}.$$

You will need to use the result

$$\int z J_0(z)\, dz = z J_1(z),$$

which is mentioned in the text.

Now write an integral formula for $u(r)$ involving Bessel functions.