Worksheet: Famous little calculations

1. Two facts about the complex exponential function are: (i) $e^{i\theta} = \cos\theta + i\sin\theta$, and (ii) $(e^z)^w = e^{zw}$. Using these facts, compute $e^{i2\theta}$ two different ways:

$$= e^{i2\theta} =$$

Because the two expansions are equal, thereby derive these trigonometric identities:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta, \qquad \sin(2\theta) = 2\cos\theta\sin\theta$$

2. (*I hope you have seen this before, but it may have been a while. A reminder is harmless.*) Let $x = 0.5757575757\cdots = 0.\overline{57}$. By computing $100x$, then subtracting x, and then cancelling a lot of digits, show that

$$x = \frac{57}{99} = \frac{19}{33}.$$

3. Let r be an complex number not equal to 1, and let a be any complex number. Show that

$$a + ar + ar^2 + ar^3 + \cdots + ar^n = a\frac{1 - r^{n+1}}{1 - r}.$$

Hint: Let S be the left side. Apply problem **2** logic. Again, you may have seen this before.