Review Topics for in-class Midterm Exam II on *Wednesday 6 April, 2016*

Midterm Exam II will cover these sections in Gamelin, *Complex Analysis*:

I.6, II.5, II.6, II.7, III.1, III.2, III.3, III.4, IV.1, IV.2

You may bring 1/2 of a sheet of paper,[1] to the exam, as notes, with anything you want written on it.

The problems will be of these types: state definitions, state theorems, describe or illustrate basic geometrical ideas, make basic applications of theorems, make basic calculations, prove simple theorems. In everything below, $z = x + iy$ is a complex number and $f(z) = u(x, y) + iv(x, y)$ is a complex-valued function.

Key Background. These topics from Midterm I material are essential for success:

- definition of *analytic* for $f(z)$
- polar form of complex number: $z = re^{i\theta}$
- definitions of e^z and $\operatorname{Arg} z$
  - $|e^z| = e^x$ and $-\pi < \operatorname{Arg} z \leq \pi$
- Cauchy-Riemann equations
  - $f(z)$ is analytic if and only if u, v satisfy the Cauchy-Riemann equations

Definitions. Know how to define:

- $\operatorname{Log} z$ and $\log z$
- the *Laplacian* of a function $u(x, y)$
- $u(x, y)$ is *harmonic*
- $v(x, y)$ is the *harmonic conjugate* of $u(x, y)$
- $f(z)$ is *conformal*
- $f(z)$ is a *fractional linear transformation* (or *Möbius transformation*)
- *translations*, *dilations*, *inversion*
- as adjectives describing paths: *closed*, *smooth*, *piecewise-smooth*
- the *line (path) integral* $\int_\gamma P\,dx + Q\,dy$
- *positively-oriented* boundary ∂D
- *exact* differential $P\,dx + Q\,dy$ (p. 76)
- *closed* differential $P\,dx + Q\,dy$ (p. 78)
- line integral $\int_\gamma P\,dx + Q\,dy$ is *independent of path* (p. 77)
- $F(z)$ is a *primitive* for $f(z)$

Theorems. Understand, and be able to use as facts, these theorems. Be able to prove if marked.

- Theorem (p. 55). If $f(z)$ is analytic then u, v are harmonic. [**Be able to prove.**]
- Theorem (p. 56). Conditions under which $u(x, y)$ has a harmonic conjugate.

[1] Letter paper, i.e. 8.5×11 inches.

- Theorem (p. 59). If $f(z)$ is analytic and $f'(z_0) \neq 0$ then f is conformal at z_0.
- Theorem (p. 65). Every Möbius transformation is the composition of dilations, translations, and an inversion. **[Be able to prove.]**
- Theorem (p. 65). Every Möbius transformation maps circles/lines to circles/lines.
- Green's Theorem (p. 73).
- Fundamental Theorem of Calculus for exact differentials dh (p. 76).
- Lemma (p. 78). Exact differentials are closed. **[Be able to prove.]**
- Theorem (p. 79). Conditions under which a closed differential is exact.
- Theorem (p. 81, top). Conditions under which a line integral is independent of path.
- *meta*-Theorem (p. 82): independent of path $\iff$ exact $\implies$ closed.
  - And conditions for when "exact $\iff$ closed".
- Theorem (p. 83). Conditions under which harmonic conjugate exists.
- Theorem (p. 85). Mean Value Property.
- Theorem (p. 105). ML-estimate.
- Fundamental Theorems of Calculus for Analytic Functions (=FTCAF; p. 107 and p. 108)

Calculations. Know how to do basic examples. **Be able to show why the formula is true in general, starting from definitions, for any of these**.

- check $u(x, y)$ is harmonic
- given $u(x, y)$, compute harmonic conjugate $v(x, y)$ if appropriate
- Möbius transformation constructions:
  - $f(z)$ sends z_0, z_1, z_2 to $0, 1, \infty$
  - given $f(z)$, construct inverse $f^{-1}(z)$
  - given $f(z), g(z)$, construct composition $h(z) = f(g(z))$
  - $f(z)$ sends z_0, z_1, z_2 to w_0, w_1, w_2
- parameterize circlular, rectangular, or triangular paths
- take a parameterized curve $\gamma(t) = (x(t), y(t))$, on $a \leq t \leq b$, along with given functions $P(x, y)$, $Q(x, y)$, and convert the integral $\int_\gamma P\,dx + Q\,dy$ into a concrete integral over t
- apply Green's theorem to compute $\int_{\partial D} P\,dx + Q\,dy$
- given closed differential $P\,dx + Q\,dx$ on appropriate domain (p. 79), construct $h(x, y)$ so that $dh = P\,dx + Q\,dx$
- given curve γ, compute integral $\int_\gamma f(z)\,dz$ with or without FTCAF
- given curve γ, compute integral $\int_\gamma f(z)\,|dz|$
- compute ML-estimate for $\left|\int_\gamma f(z)\,dz\right|$

Sketchables/Describeables. Know how to show, or describe in clear words, some basic examples. *Regarding sketches: Make it adequately big! Give at least one indication of scale on each axis.*

- image of sets under $\operatorname{Log} z$
- image of grids (cartesian or polar) under *simple* analytic $f(z)$, respecting conformal-ness
- given domain D described by words or inequalities, sketch ∂D with positive orientation