Review Topics for the in-class Final Exam, 8–10 am on *Thursday 5 May, 2016*

The Final Exam will cover *all of the sections already covered on Midterms I and II*, plus these sections, from Gamelin, *Complex Analysis*:

IV.3, IV.4, IV.5, IV.6, V.1, V.2, V.3, V.4

The above-listed sections of recent material will be the focus of the Final Exam. However, you must also see the Review Topics handouts for Midterms I and II. Everything on those handouts is "fair game" for the Final Exam. The "Key Background" below lists the most-likely-to-be-used material from the Midterms.

You may bring only *one entire* sheet of paper—letter paper, i.e. 8.5×11 inches—to the exam, as notes, with anything you want written on it. No books, smart phones, or calculators.

The problems will be of these types: state definitions, state theorems, describe or illustrate basic geometrical ideas, make basic applications of theorems, make basic calculations, prove simple theorems. In everything below, $z = x + iy$ is a complex number and $f(z) = u(x, y) + iv(x, y)$ is a complex-valued function.

Key Background. These topics from Midterm I and II material are essential for success:

- definition of *analytic* for $f(z)$
- polar form of complex number: $z = re^{i\theta}$
- definitions and basic properties of e^z, $\operatorname{Arg} z$, and $\operatorname{Log} z$
- Cauchy-Riemann equations
- if $f(z)$ is analytic then u and v are harmonic
- Green's Theorem
- the "ML" estimate (upper bound) of an integral

Definitions. Know how to define:

- an *entire* function $f(z)$ (p. 118)
- a series $\sum a_k$ *converges* (p. 130)
- a series $\sum a_k$ *converges absolutely* (p. 131)
- a sequence of functions $f_k(z)$ *converges pointwise* to $f(z)$ (p. 133)
- a sequence of functions $f_k(z)$ *converges uniformly* to $f(z)$ (p. 134)
- a series of functions $\sum g_j(z)$ *converges pointwise* or *uniformly* (p. 135)
- the *radius of convergence* $0 \le R \le \infty$ of a power series $\sum a_k(z - z_0)^k$ (p. 138)

Theorems. Understand, and be able to use as facts, these theorems. Be able to prove if marked.

- Cauchy's Theorem (p. 110). [**Be able to prove as follows:** $\int_{\partial D} f(z)\,dz = \int_{\partial D}(u+iv)(dx+i\,dy) = \int_{\partial D} u\,dx - v\,dy + i\int_{\partial D} v\,dx + u\,dy$ **and Green's Theorem and Cauchy-Riemann equations.**]

- Cauchy Integral Formulas: (4.1) on p. 113, (4.2) on p. 114.
- Cauchy Estimates (p. 118). **[Be able to prove from Cauchy Integral Formula (4.2) and ML estimate.]**
- Liouville's Theorem (p. 118). **[Be able to prove.]**
- Fundamental Theorem of algebra (p. 4 in section I.1; proven p. 118).
- Morera's Theorem (p. 199; understand why this is a converse of Cauchy's Theorem).
- Comparison Test (p. 131).
- Theorem: *If* $\sum a_k$ converges *then* $a_k \to 0$ (p. 131).
- Theorem: If $\sum a_k$ converges absolutely then $\sum a_k$ converges (p. 131).
- Theorem: If $f_k(z) \to f(z)$ uniformly then
  - $f_j(z)$ continuous $\implies$ $f(z)$ continuous (p. 134)
  - $\int_\gamma f_j(z)\,dz \to \int_\gamma f(z)\,dz$ (p. 135)
  - $f_j(z)$ analytic $\implies$ $f(z)$ analytic (p. 136)
- Weierstrauss M-Test (p. 135).
- Theorem that there is a well-defined radius of convergence $0 \le R \le \infty$ (p. 138).
- Theorem giving Taylor's formula for a_k by differentiation: $a_k = \frac{f^{(k)}(z_0)}{k!}$ (p. 140). **[Be able to prove.]**
- Ratio Test to determine R (p. 141).
- Root Test to determine R (p. 142).
- Theorem which shows that if $f(z)$ is analytic in a disc around z_0 then it has a power series representation (p. 144).
  - And you can calculate a_k by integration (4.3): $a_k = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,d\zeta$
  - And there is an upper bound (4.4): $|a_k| \le \frac{M}{r^k}$
- Corollary that the radius of convergence R is the maximum of radii of discs around z_0 on which $f(z)$ is analytic (p. 146).

Calculations. Know how to do basic examples.

- Answer *any* question about a geometric series $\sum_{k=0}^{n} ar^n$ or $\sum_{k=0}^{\infty} ar^n$.
- Given power series $\sum a_k(z - z_0)$, find radius of convergence R by the Ratio or Root Tests.
- Given an analytic function $f(z)$ and a point z_0, find its power series by using Taylor's formula for a_k (p. 140).
- Given an analytic function $f(z)$ and a point z_0, find the radius of convergence R of its power series by using the Corollary on p. 146.

Slogans. Understand what I am saying here. Find the page where there is the best support for, or a rigorous version of, these slogans, and understand that statement.

- All integrals over closed curves are zero, if the integrand is analytic.
- Integrals can be used to compute derivatives of analytic functions.
- Any analytic function can be expanded in power series.
- The radius of convergence of a power series is merely the distance to the first singularity.