Assignment #8

Due Monday, 11 April 2016

Please read Sections IV.3, IV.4, and IV.5 in the textbook. I don't really like the book's Exercises for IV.3, so I wrote Problems P4 and P5 which do things in more understandable steps. I will grade the circled Exercises and all the Problems below.

Section IV.4, page(s) 116–117, Exercises:

1 (a) (c) (e) (g) (*Please show enough work to make it clear how you have used the Cauchy Integral Theorem. Especially: What is the $f(z)$ you used? Did you use (4.1) or (4.2)?*)

1 (b) (d) (f) (h)

2 (*Show: If the function $u(x, y)$ is harmonic in a small disk D around $z_0 = (x_0, y_0)$ then $u(x, y)$ has partial derivatives of all orders. Hint: Harmonic conjugate.*)

Section IV.5, page(s) 119, Exercises:

1

2

Problem P4. **(a)** Show, by real-valued calculus methods, that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}.$$

(*Hint*. There is a famous trick here, namely that you compute the *square* of what you want, by *using polar coordinates*.)

(b) Let $t \in \mathbb{R}$ be fixed. Let $R > 0$. Let D be the rectangle with vertices $\pm R$ and $it \pm R$. Compute

$$\int_{\partial D} e^{-z^2/2}\,dz.$$

(*Hint*. Easy. Justify with the name of a theorem.)

(c) Expand the integral in part **(b)** into four integrals along each of the sides of the rectangle D. Use part **(a)** to find the limit, as $R \to \infty$, of one of these integrals.

(d) Two of the other integrals can be shown to have limits of zero as $R \to \infty$. Show this by using the "ML-estimate" theorem in section IV.1.

(e) Conclude from the above parts that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, e^{-itx}\, dx = e^{-t^2/2}.$$

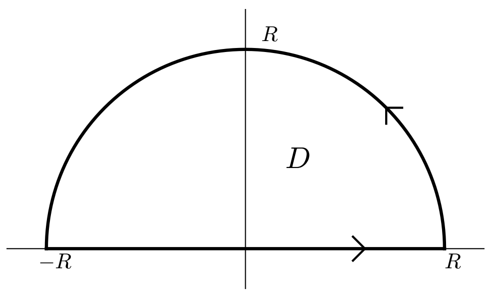
Comment. What have you accomplished? The *Fourier transform* of a function $f(x)$ defined on the real line is the new function

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-itx}\, dx.$$

(Other choices for defining the Fourier transform are possible, as the "2π" can move around.) You have shown that if $f(x) = e^{-x^2/2}$ then $\hat{f}(t) = f(t)$, that is, *this* f is its own Fourier transform.

This fact arises in quantum mechanics as the statement that the ground state of the harmonic oscillator is a minimum-uncertainty state. That is, $f(x) = e^{-x^2/2}$ is the ground state, in that case, and in general the Fourier transform of the position state is the momentum state.

Problem P5. Let $R > 1$. Consider the positively-oriented semi-circular curve below, a closed curve which is the boundary of a half disk D.



(a) Use Cauchy's Theorem to show that

$$\int_{\partial D} \frac{1}{z^2+1}\, dz = \frac{1}{2i} \int_{\partial D} \frac{1}{z-i}\, dz.$$

(*Hint*. Factor z^2+1. Then use partial fraction decomposition.)

(b) By deforming the contour to the boundary of a small disk centered at i, show that

$$\int_{\partial D} \frac{1}{z-i}\, dz = 2\pi i.$$

(c) By using the "ML-estimate" theorem, take the $R \to \infty$ limit of the integral in part **(a)**, and use part **(b)**, to conclude that

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx = \pi.$$

(d) The result in part **(c)** can be checked by standard calculus methods. Do so.